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Twistor Transform

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Abstract. We review the foundations of twistor theory, with the aim of expressing the Penrose integral transforms in the language of sheaf cohomology. The key vocabulary of sheaves and fibre bundles is developed in detail, enabling a formal discussion of gauge theories. We present a rigorous analysis of spinor notation and formulate the zero-rest-mass free field equations. Proofs of the Penrose and Penrose-Ward transformations are sketched and physically relevant examples are calculated explicitly.

1 Introduction

"*Le plus court chemin entre deux vérités dans le domaine réel passe par le domaine complexe.*" —Jacques Hadamard

In the forty years since its inception, twistor theory has found applications in many areas of mathematics. Early research centred around its potential as a quantum theory of spacetime. Yet despite major progress, twistors are yet to have a major impact on fundamental physics. Indeed twistor techniques and their generalizations have had much greater success in integrable systems and differential geometry.

Twistor transforms are perhaps the most potent tool provided by the twistor programme. The simplest are integral transforms which enable the automatic solution of classes of equations. The original Penrose transform has this form, solving zero rest mass field equations on Minkowski space. More advanced twistor transforms relate fields to vector bundles. These yield new perspectives on gauge theory, instantons and monopoles.

To fully appreciate the power of the twistor transform requires some considerable machinery. We must study sheaf cohomology and fibre bundles, familiar to algebraic geometers. We need spinor notation and field theory employed by theoretical physicists. Finally we should follow the pioneering Penrose into the world of twistor geometry.

These daunting prerequisites obscure our goal. Therefore it is pedagogically important to compute a few simple examples before we set off. The reader should refer back to these for motivation in the mathematically denser sections of the text.

1.1 Motivational Examples

Consider a flat 4-dimensional manifold M with metric η of definite signature. The **wave equation** for a scalar field φ takes the form

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\varphi = 0$$

We aim to solve this equation in neutral signature and Lorentzian signature using an integral transform technique, somewhat like a Fourier tranform.

We start with the neutral signature case, which can be solved by a **John transform** as follows. Let $T = \mathbb{R}^3$ and $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be an arbitrary smooth function. Let M be the space of oriented lines in T, with typical element

$$\ell(\mathbf{u}, \mathbf{v}) = \{\mathbf{v} + t\mathbf{u} : t \in \mathbb{R}\}$$

for some $|\mathbf{u}| = 1$ with $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$. Consider the tangent bundle of the 2-sphere

$$TS^2 = \{(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^3 \times \mathbb{R}^3 : |\mathbf{u}| = 1 \text{ and } (\mathbf{u}, \mathbf{v}) = 0\}$$

where $(\mathbf{u}, \mathbf{v})$ denotes the Euclidean inner product. Define a bijection

$$\begin{aligned} M &\longrightarrow TS^2 \\ \ell(\mathbf{u}, \mathbf{v}) &\longmapsto (\mathbf{u}, \mathbf{v} - (\mathbf{v}, \mathbf{u})\mathbf{u}) \end{aligned}$$

where the second component is the point on $\ell(\mathbf{u}, \mathbf{v})$ closest to the origin. Hence we may *locally* identify M with $\mathbb{R}^4$.

Choose local coordinates (t, x, y, z) for M, writing

$$\ell = \{(t + sy, x + sz, s) : s \in \mathbb{R}\}$$

These parameterize all lines which do not lie in planes of constant x_3. Now define a function φ on M by

$$\varphi(\ell) = \int_\ell f$$

which reads in coordinates

$$\varphi(t, x, y, z) = \int_{-\infty}^{\infty} f(t + sy, x + sz, s) ds$$

Now there are 4 parameters and f is defined on $\mathbb{R}^3$ so we expect a condition on φ. Differentiating under the integral sign we obtain the wave equation

$$\frac{\partial^2 \varphi}{\partial t \partial z} - \frac{\partial^2 \varphi}{\partial x \partial y} = 0$$

It is natural to ask whether this construction can be inverted. Indeed John [25] showed that every solution of the wave equation can be obtained from some f.

This preliminary example demonstrates a defining philosophy of twistor theory. Namely, an unconstrained function on 'twistor space' T yields the solution to a differential equation on 'Minkowski space' M, via an integral transform. We also have a simple geometrical correspondence, another characteristic feature of twistor methods. Specifically we see

$$\begin{aligned} T &\longleftrightarrow M \\ \{\text{point in } T\} &\longrightarrow \{\text{oriented lines through point}\} \\ \{\text{line in } T\} &\longleftarrow \{\text{point in } M\} \end{aligned}$$

For the Lorentzian signature case we employ the **Penrose transform**. Let

$T = \mathbb{P}^3$ and $f : \mathbb{P}^3 \longrightarrow \mathbb{C}$ be holomorphic except for finitely many poles on any restriction to $\mathbb{P}^1 \subset \mathbb{P}^3$. Let M be Minkowski space, with coordinate (t, x, y, z) and define a function φ on M by

$$\varphi(t, x, y, z) = \frac{1}{2\pi i} \oint_\Gamma f(-(t + ix) + \lambda(z - y), (z + y) + \lambda(-t + ix), \lambda) d\lambda$$

where Γ is any closed contour in $\mathbb{P}^1$ which avoids the poles of f. Again we expect one condition on φ and differentiating under the integral gives the wave equation

$$\frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial y^2} - \frac{\partial^2 \varphi}{\partial z^2} = 0$$

The Penrose transform is more sophiscated than the John transform, since it involves contour integration over a complex space. In particular, note that we may change the contour Γ or add a holomorphic function to f without changing φ. Thus to define an inverse transform we need to consider equivalence classes of functions and contours. Mathematically these are described by sheaf cohomology, which is the subject of §2.

Be warned that our notation in this section was deliberately imprecise. The knowledgeable reader will notice that we have failed to distinguish between twistor space and its projectivisation. In §5 and §6 we shall reformulate our language rigorously. For the purposes of these examples, the notation abuse is warranted to maintain transparency.

1.2 Outline

This review is split into three sections. In §2 and §3 we introduce the pure mathematical background underpinning the field. These topics may appear esoteric at first, but are of vital importance to modern mathematics far beyond twistor theory. We also precisely formulate the notion of a gauge theory, explaining oft-quoted results in a natural way.

The material in §4 and §5 is of a different flavour. Here we introduce notational conventions ubiquitous in twistor theory, but perhaps lesser known outside the field. We study twistors from several different perspectives, leaving the most formal arguments until last. The interplay between geometry and physics guides our journey through the twistor landscape.

Finally we amalgamate all our earlier ideas in §6. We meet twistor transforms in several related incarnations, observing how they solve physically important equations. This section is less detailed and more fast-paced than the main body of the text, and is intended to whet the reader's appetite for a serious study of relevant papers.

We have adopted a formal style, more familiar to pure mathematicians than theoretical physicists. This distinguishes our review from other treatments of the subject. We hope that the added clarity and rigour of our work will enable readers to swiftly develop a deep understanding of the central concepts. A healthy portion of examples and remarks is provided throughout the text, helping to maintain intuitive appeal.

We use the following notation throughout

$$
\begin{aligned}
\eta &= \text{Minkowski metric, signature } +--- \\
M &= \text{Minkowski space } \mathbb{R}^4 \text{ equipped with metric } \eta \\
\mathbb{C}M = \mathsf{M}^I &= \text{complexified Minkowski space } \mathbb{C}^4 \\
M^c &= \text{conformally compactified Minkowski space} \\
\mathbb{C}M^c = \mathsf{M} &= \text{complexified conformally compactified Minkowski space} \\
T &= \text{twistor space } \mathbb{C}^4 \text{ equipped with Hermitian form } \Sigma \\
\mathbb{P}T = \mathsf{P} &= \text{projective twistor space } \mathbb{P}^3
\end{aligned}
$$

1.3 Principal References

I am primarily indebted Huggett and Tod [21], Ward and Wells [39] and Dunajski [9] whose books introduced me to the subject. Much of the material herein is based on arguments found in these volumes. Where appropriate I have added detail, or modified arguments to suit my purposes. I rarely cite these works explicitly, so I must give full credit to the authors now.

My greatest intellectual homage must be to Sir Roger Penrose. Without his imagination this beautiful branch of mathematics may have remained an unknown unknown. It is no surprise that his papers occupy almost one-sixth of the bibliography! I was fortunate enough to hear him speak to the Archimideans in February 2013 which particularly inspired me to include Example 4.37.

Finally I am extremely grateful to my supervisor, Dr. Maciej Dunajski, for the advice and encouragement I have received over the past few months. Striking out into the jungle of research mathematics is both exhilarating and terrifying. His guidance has enabled me to maximise the former and minimise the latter.

2 Sheaf Theory

We saw in §1.1 that the process of inverting a twistor transform is nontrivial in general. There is a degeneracy, or gauge freedom, in the choice of twistor function. Eastwood et al. [10] articulated the correct viewpoint. We should view the twistor transform in terms of the cohomology classes of certain sheaves. To make this precise we must first introduce the mathematical formalism of sheaf theory.

In this section we encounter the basic definitions in two different guises. First we examine the abstract language preferred by modern algebraic geometers. We connect this to the geometric picture given by étalé spaces, which is more commonly used in twistor theory. We omit the proofs of equivalence, for they amount to no more than definition chasing. We conclude with a thorough exposition of elementary sheaf cohomology, including intuitive motivations and examples often lacking in terser reviews.

Pure mathematicians should regard this section merely as a useful reference, and may freely skip it on a first reading. Theoretical physicists might also wish to defer a detailed study of the material. A full understanding is not essential until §6.

2.1 Basic Definitions

Definition 2.1. Let X be a topological space. An abelian **presheaf** $\mathcal{F}$ on X consists of

1. $\forall$ open $U \subset X$ an abelian group $\mathcal{F}(U)$
2. if $V \subset U$ open subsets of X a restriction homomorphism

$$\rho_V : \mathcal{F}(U) \longrightarrow \mathcal{F}(V),\ s \longmapsto s|_V$$

subject to the conditions

1. $\mathcal{F}(\emptyset) = \emptyset$
2. $\forall$ open U, $\mathcal{F}(U) \longrightarrow \mathcal{F}(U)$ the identity homomorphism
3. $W \subset V \subset U$ then the following diagram of restriction maps commutes

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r] \arrow[dr] & \mathcal{F}(V) \arrow[d] \\
 & \mathcal{F}(W)
\end{tikzcd}$$

An element $s \in \mathcal{F}(U)$ is called a **section** of $\mathcal{F}$ over U. $s \in \mathcal{F}(X)$ is called a **global** section.

Definition 2.2. A presheaf is called a **sheaf** iff $\forall$ open $U \subset X$ if $U = \bigcup U_i$ open cover and we are given $s_i \in \mathcal{F}(U_i)$ with $s_i|_{U_i \cup U_j} = s_j|_{U_i \cup U_j}$ then $\exists$ a unique $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i \ \forall\ i$.

Remark 2.3. One can intuitively view a sheaf as a democratic presheaf; that is, a presheaf on which global data is completely determined by local data.

Example 2.4. Let X be a complex manifold. Define sheaves $\mathcal{O}$, $\mathcal{O}^*$, Λ^p and $\mathbb{Z}$

$$\begin{aligned}
\mathcal{O}(U) &= \{\text{holomorphic } s : U \longrightarrow \mathbb{C} \text{ under addition}\} \\
\mathcal{O}^*(U) &= \{\text{nonzero holomorphic } s : U \longrightarrow \mathbb{C} \text{ under multiplication}\} \\
\Lambda^p(U) &= \{\text{differential } p\text{-forms on } U \text{ under addition}\} \\
\mathbb{Z}(U) &= \{\text{constant } s : U \longrightarrow \mathbb{Z} \text{ under addition}\}
\end{aligned}$$

Definition 2.5. The **stalk** of a presheaf at $x \in X$ is defined to be the group

$$\mathcal{F}_x = \{(U, s) : U \ni x,\ s \in \mathcal{F}(U)\}/\sim$$

where $(U, s) \sim (V, t)$ iff $\exists\ W \subset U \cap V$, $W \ni x$ such that $s|_W = t|_W$. An element of $\mathcal{F}_x$ is called a **germ**. We denote a germ at x by $[U, s]$ or $[s, x]$.

Remark 2.6. The stalk encodes the behaviour of sections in an infinitesimal region around x.

Example 2.7. Let $\mathcal{O}$ be the sheaf of holomorphic functions on $\mathbb{C}$. Then the stalk at x is the ring of power series convergent in some neighbourhood of x.

Definition 2.8. A presheaf $\mathcal{G}$ is a **subpresheaf** of $\mathcal{F}$ if $\mathcal{G}(U)$ is a subgroup of $\mathcal{F}(U)$ for all U and the restriction maps of $\mathcal{G}$ are induced from those of $\mathcal{F}$.

Definition 2.9. Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves on X. A **morphism** $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ is a collection of homomorphisms $\varphi_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ for all $U \subset X$ open, such that whenever $V \subset U$ we have $\rho_V \circ \varphi_U = \varphi_V \circ \rho_U$. An **isomorphism** is a morphism with a two-sided inverse.

Remark 2.10. Observe that φ induces a homomorphism $\varphi_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x$ on stalks, explicitly given by $\varphi_x : [U, s] \longmapsto [U, \varphi_U(s)]$.

Definition 2.11. Let $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of presheaves. The **presheaf kernel** of φ is defined by

$$\ker^{\text{pre}}(\varphi)(U) = \ker(\varphi_U)$$

The **presheaf image** of φ is defined by

$$\mathrm{im}^{\mathrm{pre}}(\varphi)(U) = \mathrm{im}(\varphi_U)$$

Clearly these are subpresheaves of $\mathcal{F}$ and $\mathcal{G}$ respectively.

Lemma 2.12. Let $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaves. Then $\ker^{\mathrm{pre}}(\varphi)$ is a sheaf.

Proof. Let $U \subset X$ with $U = \bigcup_i U_i$ and $s_i \in \ker^{\mathrm{pre}}(\varphi)(U_i)$. Suppose also that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_i}$. Since $\mathcal{F}$ a sheaf there certainly exists $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$. Now note that $\varphi(s)|_{U_i} = \varphi(s|_{U_i}) = \varphi(s_i) = 0$ by definition of morphism. Since $\mathcal{G}$ is a sheaf also, we must have $\varphi(s) = 0$, whence $s \in \ker^{\mathrm{pre}}(\varphi)(U)$. $\square$

Remark 2.13. Note that $\mathrm{im}^{\mathrm{pre}}(\varphi)$ is not a sheaf in general. Indeed let $p \neq q \in \mathbb{R}$ and define a sheaf $\mathcal{G}$ on $\mathbb{R}$ by

$$\mathcal{G}(U) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } \{p, q\} \subset U \\ \mathbb{Z} & \text{if } p \in U \text{ and } q \notin U \\ \mathbb{Z} & \text{if } p \notin U \text{ and } q \in U \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathcal{F}$ be the constant sheaf $\mathbb{Z}$. Define a natural morphism $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ by

$$\varphi_U = \begin{cases} \text{diagonal} & \text{if } \{p, q\} \subset U \\ \text{identity} & \text{if } p \in U \text{ and } q \notin U \\ \text{identity} & \text{if } p \notin U \text{ and } q \in U \\ \text{zero} & \text{otherwise} \end{cases}$$

Now take $X = U_1 \cup U_2$ with $p \in U_1$, $q \notin U_1$ and $p \notin U_2, q \in U_2$. Choose $s_1 \in \mathcal{G}(U_1)$ to have $s_1(x) = a \in \mathbb{Z}$ and $s_2 \in \mathcal{G}(U_2)$ to have $s_2(x) = b \neq a \in \mathbb{Z}$. Since $p, q \notin U_1 \cap U_2$ we see that s_1 and s_2 automatically agree on the overlap. Now defining $s \in \mathcal{G}(X)$ by $s(x) = (a, b)$ we see that $s|_{U_1} = s_1$ and $s|_{U_2} = s_2$. It is now clear that $s \notin \mathrm{im}^{\mathrm{pre}}(\varphi)(X)$.

This motivates the following definition, which might seem somewhat arcane at first glance.

Definition 2.14. Let $\mathcal{F}$ be a presheaf on X. The **associated sheaf** $\mathcal{F}^+$ on X is the set of functions $s : U \longrightarrow \bigsqcup_{x \in U} \mathcal{F}_x$ such that

1. For all $x \in U$, $s(x) \in \mathcal{F}_x$.
2. For all $x \in U$, there exists $W \ni x$ with $W \subset U$ and an element $t \in \mathcal{F}(W)$ such that for all $y \in W$, $s(y) = [W, t]$.

Remark 2.15. In fact, this is a very concrete construction. The procedure first identifies the sections of $\mathcal{F}$ which have the same restriction, and then adds in all sections which can be patched together.

Definition 2.16. Let $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaves. The **kernel** of φ is defined by

$$\ker(\varphi) = \ker^{\text{pre}}(\varphi)$$

The **image** of φ is defined by

$$\text{im}(\varphi) = (\text{im}^{\text{pre}}(\varphi))^{+}$$

We say that φ is **injective** if $\ker(\varphi) = 0$ and **surjective** if $\text{im}(\varphi) = \mathcal{G}$.

Definition 2.17. Let $\mathcal{G}$ be a subsheaf of $\mathcal{F}$. The **quotient sheaf** $\mathcal{F}/\mathcal{G}$ is the sheaf associated to the presheaf $(\mathcal{F}/\mathcal{G})^{\text{pre}}(U) = \mathcal{F}(U)/\mathcal{G}(U)$.

2.2 Etalé Spaces

Definition 2.18. We define the **étalé space** of a presheaf $\mathcal{F}$ on X to be the set $\mathcal{F}X = \bigsqcup_{x \in X} \mathcal{F}_x$. There is a natural projection map $\pi : \mathcal{F}X \longrightarrow X$ taking $(U, s) \in \mathcal{F}_x$ to x. For each open $U \subset X$ and section $s \in \mathcal{F}(U)$ we define an **associated map** $\bar{s} : U \longrightarrow \mathcal{F}X$ by $x \longmapsto s_x$, the germ of s at x. Clearly $\pi \circ \bar{s} = \text{id}$ so $\bar{s}$ is a section of π in the sense of Definition 3.4. We endow $\mathcal{F}X$ with the largest topology such that the associated maps $\bar{s}$ are continuous $\forall\ s \in \mathcal{F}(U),\ \forall$ open $U \subset X$.

Lemma 2.19. $\mathcal{F}$ is a sheaf over X iff for each open $U \subset X$ every continuous section of π over U is the associated map for some $s \in \mathcal{F}(U)$.

Remark 2.20. We therefore immediately note that the set of continuous sections of a fibre bundle is automatically a sheaf over the base space. Such sheaves play a vital role in §6.

Remark 2.21. We may now articulate a more geometrical definition of the associated sheaf. The sheaf associated to a presheaf $\mathcal{F}$ is given by the sheaf of continuous sections of its étalé space $\mathcal{F}X$.

Lemma 2.22. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X. A morphism of sheaves is equivalently a continuous map $\varphi : \mathcal{F}X \longrightarrow \mathcal{G}X$ which preserves fibres and is a group homomorphism on each fibre.

Lemma 2.23. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves over X, and $\varphi : \mathcal{F}X \longrightarrow \mathcal{G}X$ be a sheaf morphism. Then we may identify the kernel and image of φ as

$$\ker(\varphi) = \{s \in \mathcal{F} : \varphi(s) = 0 \in \mathcal{G}_x \text{ if } s \in \mathcal{F}_x\}$$
$$\operatorname{im}(\varphi) = \{t \in \mathcal{G} : t = \varphi(s) \text{ for some } s \in \mathcal{F}\}$$

Definition 2.24. A sequence of maps between spaces

$$G_0 \xrightarrow{f_0} G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} \dots$$

is called **exact** if $\operatorname{im}(f_i) = \ker(f_{i+1}) \ \forall \ i \geq 0$.

Theorem 2.25. A sequence of sheaves over X and sheaf morphisms is exact iff the corresponding sequence of stalks and group homomorphisms is exact at all $x \in X$.

Proof. Immediate from the étalé space perspective. $\square$

Remark 2.26. Invoking this lemma is a convenient way to prove exactness for sequences of sheaves. We find it extremely useful in §6.

Example 2.27 (The Exponential Sheaf Sequence). We define a short exact sequence of sheaves on a complex manifold X by

$$0 \to \mathbb{Z} \xrightarrow{i} \mathcal{O} \xrightarrow{e} \mathcal{O}^* \to 0$$

where i is the inclusion map and e is defined by

$$e_U(f) = \exp(2\pi i f) \text{ for } f \in \mathcal{O}(U)$$

The only nontrivial part of exactness is the surjectivity of e. It suffices to verify this on stalks. Let $[g, z]$ be the germ of a nonzero holomorphic function at z. Choose some simply connected neighbourhood $U \ni z$ on which $g \neq 0$. Then we may define a holomorphic branch of $\log(g)$ on U by fixing $z_0 \in U$ and setting

$$\log(g)(z) = \log(g(z_0)) + \int_{z_0}^{z} \frac{dg}{g}$$

Now choosing $f = \frac{1}{2\pi i} \log(g)$ we see that $e_x([f, z]) = [g, z]$ as required. It is interesting to ask whether the sequence of global sections

$$0 \to \mathbb{Z}(X) \xrightarrow{i} \mathcal{O}(X) \xrightarrow{e} \mathcal{O}^*(X) \to 0$$

is exact. Once again the surjectivity of e is the only problem. To analyse the obstruction to the sequence being exact we must introduce the methods of cohomology theory.

2.3 Čech Cohomology

Definition 2.28. Let X be a topological space $\mathcal{U} = (U_i)_{i \in I}$ an open covering with I some fixed, well-ordered index set. Let $\mathcal{F}$ be an abelian sheaf on X. For any finite set $i_0, \dots i_p \in I$ we denote $U_{i_0} \cap \dots \cap U_{i_p} = U_{i_0, \dots i_p}$. For $p \geq 0$ we define the p^{th} **cochain group** of $\mathcal{F}$ with respect to $\mathcal{U}$ by

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < \dots < i_p} \mathcal{F}(U_{i_0, \dots i_p})$$

An element of the cochain group is called a **cochain** and comprises a collection of sections $\alpha_{i_0, \dots i_p} \in \mathcal{F}(U_{i_0, \dots i_p})$ for every ordered $(p+1)$-tuple of elements of I. We define the **coboundary map** $d : C^p \longrightarrow C^{p+1}$ by

$$(d\alpha)_{i_0, \dots i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \rho_{i_0, \dots i_{p+1}} \alpha_{i_0, \dots \hat{i_k}, \dots i_{p+1}}$$

where the hat symbol denotes the omission of an index, and ρ denotes restriction. It is clear that $d^2 = 0$ so $(C^\bullet, d)$ defines a complex of abelian groups, called the **Čech complex**. We define the p-th **cocycle group** and the p-th **coboundary group** by

$$Z^p = \ker(d : C^p \longrightarrow C^{p+1})$$
$$B^p = \operatorname{im}(d : C^{p-1} \longrightarrow C^p)$$

with elements called **cocycles** and **coboundaries** respectively. The p-th **cohomology group** measures the failure of the sequence defined by d to be exact at C^p, and is explicitly

$$H^p(\mathcal{U}, \mathcal{F}) = Z^p / B^p$$

Remark 2.29. Thus far our construction has depended upon the choice of open cover $\mathcal{U}$ for X. We pass to a covering independent notion via the method of Morrow and Kodaira [26, §2.2], introducing refinements. One can make a more direct definition, using the derived functor approach of Hartshorne [17, §III.2]. However, this approach is esoteric and practically useless for calculations, so we avoid it.

Remark 2.30. Intuitively we can view sheaf cohomology as a measure of how many more sections we obtain as we focus more locally on the base space. In this sense sheaf cohomology is a quantitative approach to determining obstructions to patching sections together.

Example 2.31. Following Ward and Wells [39, p. 176], we note without proof that the cohomology of a constant sheaf $\mathbb{F}$ on a topological space X coincides with the ordinary cohomology of X with coefficients in $\mathbb{F}$. Henceforth we freely quote results from algebraic topology in connection with this observation.

Example 2.32 (The Mittag Leffler Problem). Following Griffiths and Harris [16, p. 34] we briefly examine a motivational application of Čech cohomology. Suppose we are a discrete set of points $\{p_j\} \subset \mathbb{C}$ and asked to define a function f on $\mathbb{C}$ holomorphic on $\mathbb{C} \setminus \{p_j\}$ and with a pole of order m_j at each p_j. This is obviously trivial in any compact subset U of $\mathbb{C}$, since U necessarily contains only finitely many of the $\{p_j\}$ so we define

$$f_U(z) = \prod_{p_j \in U} (z - p_j)^{-m_j}$$

Globally, however, this process might not converge. We can nevertheless prove that the construction is possible by appealing to cohomology. Let $\{U_i\}$ be an open cover of $\mathbb{C}$ with each U_i containing at most one of the p_j. Let f_i be a meromorphic function solving the problem in U_i, and define $f_{ij} = f_i - f_j$ on U_{ij}. On U_{ijk} we automatically have $f_{ij} + f_{jk} + f_{ki} = 0$, so

$$\{f_{ij}\} \in Z^1(\{U_i\}, \mathcal{O})$$

Now solving the problem globally is equivalent to finding $g_i \in \mathcal{O}(U_i)$ such that $f_{ij} = g_j - g_i$ on U_{ij}. Indeed suppose we had such g_i. Defining $h_i = g_i + f_i \in \mathcal{O}(U_i)$ we see that h_i solves the problem locally. Moreover $h_i - h_j = 0$ on U_{ij} so the h_i extend globally. The converse is similarly trivial. Now

$$\{f_{ij} : f_{ij} = g_j - g_i\} = B^1(\{U_i\}, \mathcal{O})$$

Hence the obstruction to solving the problem is measured by $H^1(\{U_i\}, \mathcal{O})$. We see shortly that $H^1(\mathbb{C}, \mathcal{O}) = 0$ since $\mathbb{C}$ is a Stein manifold. Therefore the Mittag-Leffler problem can be solved.

Lemma 2.33. $H^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$, the set of global sections.

Proof. H^0 is the kernel of d^0, which is precisely the group of all local sections s_i which agree on intersections. But by the definition of a sheaf, this is isomorphic to the group of global sections of $\mathcal{F}$. □

Definition 2.34. An open covering $\mathcal{V} = \{V_j\}_{j \in J}$ of X is **refinement** of $\mathcal{U} = \{U_i\}_{i \in I}$ if there is a map $r : J \longrightarrow I$ such that

$$V_j \subset U_{r(j)} \ \forall \ j \in J$$

The induced map on cochains $R : C^p(\mathcal{U}, \mathcal{F}) \longrightarrow C^p(\mathcal{V}, \mathcal{F})$ is defined by

$$R(s_{i_0,\ldots i_p}) = \rho_{i_0,\ldots i_p} s_{r(i_0),\ldots r(i_p)}$$

Lemma 2.35. *R and d commute, so R defines a homomorphism $H^p(\mathcal{U}, \mathcal{F}) \longrightarrow H^p(\mathcal{V}, \mathcal{F})$.*

Proof. This is simply a tedious exercise in notation, so we omit it. □

Lemma 2.36. The homomorphism $R : H^p(\mathcal{U}, \mathcal{F}) \longrightarrow H^p(\mathcal{V}, \mathcal{F})$ depends only on $\mathcal{U}$ and $\mathcal{V}$ not on the choice of map r.

Proof. We refer the interested reader to Morrow and Kodaira [26, p. 32]. □

Definition 2.37. Let $\mathcal{F}$ be a sheaf on a space X. Then the p^{th} **cohomology group** is defined by

$$H^p(X, \mathcal{F}) = \bigsqcup_{\mathcal{U}} H^p(\mathcal{U}, \mathcal{F})/\sim$$

where the disjoint union is taken over all covers of X, and two elements $s \in H^p(\mathcal{U}, \mathcal{F})$ and $t \in H^p(\mathcal{U}', \mathcal{F})$ are equivalent if there exists a common refinement $\mathcal{V}$ such that $R(s) = R'(t)$.

Remark 2.38. The process of refinement is obviously unsatisfying from a calculational perspective. However for a fixed sheaf $\mathcal{F}$ on a fixed space X there may exist a **Leray cover** $\mathcal{U}$ such that $H^p(\mathcal{U}, \mathcal{F}) = H^p(X, \mathcal{F})\ \forall\ p \geq 0$. We can then work with this fixed cover for all computations. For our purposes such a cover always exists, as the following results guarantee.

Theorem 2.39 (Leray)**.** $\mathcal{U}$ is a Leray cover for $(X, \mathcal{F})$ if $H^q(U_{i_0,\ldots i_p}, \mathcal{F}) = 0$ for all $q > 0$ and all ordered sets $i_0 < \cdots < i_p$.

Proof. We refer the interested reader to Field [12, p. 109]. □

Remark 2.40. The subsets of X on which cohomology is required to vanish for $\mathcal{U}$ to be Leray may be small relative to the global extent of X. We can regard these subsets as cohomologically trivial building blocks, from which we construct the global cohomology theory. This is analogous to the use of cell complexes in algebraic topology, see Ward and Wells [39, p. 162].

Theorem 2.41 (Cartan's Theorem B)**.** Let $\mathcal{F}$ be a coherent analytic sheaf on a Stein manifold $\mathcal{M}$. Then $H^p(\mathcal{M}, \mathcal{F}) = 0$ for all $p > 0$.

Proof. We refer the ambitious reader to Forstnerič [14, p. 52]. □

Remark 2.42. We have deliberately avoided providing rigorous definitions for the terms used in the previous theorem. Without a deep understanding of algebraic geometry the definitions would merely seem sterile and abstract. Instead we quote some examples of Stein manifolds and coherent analytic sheaves which will suffice for this essay.

Behnke and Stein [5] showed that any connected non-compact one-dimensional complex manifold is Stein. In particular if $\{U_i\}$ is an open cover of a connected Riemann surface then every U_i is automatically Stein. It is well-known that the sheaf of holomorphic sections of a holomorphic vector bundle is a coherent analytic sheaf.

Example 2.43 (Cohomology of $\mathcal{O}$ on $\mathbb{P}^1$). Following Dunajski [9, p. 303] we claim that all holomorphic functions on $\mathbb{P}^1$ are constant. Let $f \in \mathcal{O}(\mathbb{P}^1)$. Then $|f|$ has a maximum since $\mathbb{P}^1$ compact. Let U be open and connected in $\mathbb{P}^1$ and $\varphi : U \longrightarrow V \subset \mathbb{C}$ a coordinate chart. Then $f \circ \varphi^{-1}$ is a function on a connected open subset of $\mathbb{C}$ whose modulus has a maximum, so $f \circ \varphi^{-1}$ is constant by the maximum modulus theorem, whence f is constant. So $H^0(\mathbb{P}^1, \mathcal{O}) = \mathbb{C}$.

To compute $H^1(\mathbb{P}^1, \mathcal{O})$ it suffices to use Čech cohomology with the usual open cover $U_i = \{[z_0 : z_1] \mid z_i \neq 0\}$, by the previous remark. Choose a cocycle $f_{01} \in Z_1(\{U_i\}, \mathcal{O})$, and let $z = z_0/z_1$ be a coordinate on U_0. Note that z^{-1} is a coordinate on U_1. Since U_{01} is an annulus we may Laurent expand f_{01} about 0 to obtain

$$f_{01}(z) = \sum_{n=0}^{\infty} a_n z^n - \sum_{n=1}^{\infty} b_n z^{-n}$$

We therefore define

$$f_0(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{O}(U_0)$$
$$f_1(z) = \sum_{n=1}^{\infty} b_n z^{-n} \in \mathcal{O}(U_1)$$

so that $f_{01} = f_0 - f_1$ on U_{01} whence $f_{01} \in B^1(\{U_i\}, \mathcal{O})$. Thus $H^1(\mathbb{P}^1, \mathcal{O}) = 0$.

Theorem 2.44. Let $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaves over X. Then there are induced homomorphisms $\tilde{\varphi} : C^p(X, \mathcal{F}) \longrightarrow C^p(X, \mathcal{G})$ and $\varphi^* : H^p(X, \mathcal{F}) \longrightarrow H^p(X, \mathcal{G})$.

Proof. We first fix a cover $\mathcal{U}$. Define a homomorphism $\tilde{\varphi} : C^p(\mathcal{U}, \mathcal{F}) \longrightarrow C^p(\mathcal{U}, \mathcal{G})$ by $\tilde{\varphi}(s_{i_0,\dots i_p}) = \varphi_{U_{i_0,\dots i_p}}(s_{i_0,\dots i_p})$. Note that $\tilde{\varphi}$ is compatible with refinements, in the sense that it commutes with R. Therefore it descends to a homomorphism $\tilde{\varphi} : C^p(X, \mathcal{F}) \longrightarrow C^p(X, \mathcal{G})$. Moreover $\tilde{\varphi}$ commutes with d by definition of morphism. Therefore $\tilde{\varphi}$ induces a homomorphism $\varphi^* : H^p(X, \mathcal{F}) \longrightarrow$

$H^p(X, \mathcal{G})$ as required. $\square$

Lemma 2.45. Let $0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C} \to 0$ be a short exact sequence of sheaves on X. Then the induced sequence $0 \longrightarrow C^p(X, \mathcal{A}) \xrightarrow{\tilde{\alpha}} C^p(X, \mathcal{B}) \xrightarrow{\tilde{\beta}} C_0^p(X, \mathcal{C}) \longrightarrow 0$ is exact, where $C_0^p(X, \mathcal{C}) = \mathrm{im}(\tilde{\beta})$.

Proof. We must check that for all open $U \subset X$ the sequence $0 \longrightarrow \mathcal{A}(U) \xrightarrow{\alpha_U} \mathcal{B}(U) \xrightarrow{\beta_U} \mathcal{C}(U)$ is exact. By definition, $\ker(\alpha_U) = \ker(\alpha)(U) = 0$ so the sequence is exact at $\mathcal{A}$. Now since $\ker(\alpha) = 0$ the presheaf image of $\mathcal{A}$ is $\mathcal{A}$ itself. Hence the sheaf image of α is precisely its presheaf image. Therefore $\mathrm{im}(\alpha_U) = \mathrm{im}(\alpha)(U) = \ker(\beta)(U) = \ker(\beta_U)$ so the sequence is exact at $\mathcal{B}$. $\square$

Remark 2.46. The proof of the previous theorem raises an obvious question. When is the sequence $0 \longrightarrow \mathcal{A}(X) \xrightarrow{\alpha_X} \mathcal{B}(X) \xrightarrow{\beta_X} \mathcal{C}(X) \longrightarrow 0$ exact at $\mathcal{C}(X)$? Equivalently, when can we lift global sections of $\mathcal{C}$ to global sections of $\mathcal{B}$? We demonstrate that the obstruction to lifting is encoded by $H^1(X, \mathcal{A})$.

Let $\mathcal{U} = \{U_i\}$ be an open cover of X such that β_U is surjective for all $U_i \in \mathcal{U}$. Fix some arbitrary $x \in \mathcal{C}(X)$ and lift s to $t_i \in \mathcal{B}(U_i)$. The obstruction to their gluing to yield a global section of $\mathcal{B}$ is encoded by

$$f_{ij} = t_i - t_j \in \mathcal{A}(U_{ij})$$

using the exactness of the sequence at $\mathcal{A}(X)$. By definition $d(f_{ij}) = f_{ij} + f_{jk} + f_{ki} = 0$ on U_{ijk} so $\{f_{ij}\} \in Z^1(\mathcal{U}, \mathcal{A})$. Clearly $\{f_{ij}\} = 0$ in $Z^1(\mathcal{U}, \mathcal{A})$ is a sufficient condition for s to lift globally, but it is not necessary. Indeed there was an ambiguity in choosing the t_i, for we may equally choose

$$\tilde{t}_i = t_i + \epsilon_i$$

for any $\epsilon_i \in \ker(\beta)(U_i) = \mathrm{im}(\alpha)(U_i)$ by exactness. Now we note that

$$\tilde{t}_i - \tilde{t}_j = f_{ij} + \epsilon_j - \epsilon_i$$

so there exists a compatible lift iff $\{f_{ij}\} = \{\epsilon_i\}$ for some $\epsilon_i \in \mathcal{A}(U_i)$, that is to say if $\{f_{ij}\} \in B^1(\mathcal{U}, \mathcal{A})$. Therefore β_X is surjective iff $H^1(X, \mathcal{A}) = 0$.

Lemma 2.47. Assume the setup of Lemma 2.45 and recall the definition of $C_0^p(X, \mathcal{C})$. Since d commutes with $\tilde{\beta}$ we may naturally define $H_0^p(X, \mathcal{C})$. If X is paracompact then there is an isomorphism $H_0^p(X, \mathcal{C}) \cong H^p(X, \mathcal{C})$.

Proof. A version of this lemma is proved in Hirzebruch [18, §2.9]. $\square$

Theorem 2.48 (The Long Exact Sequence In Cohomology)**.** Let

$$0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C} \to 0$$

be a short exact sequence of sheaves on X. Then there is a long exact sequence in cohomology

$$0 \longrightarrow H^0(X,\mathcal{A}) \xrightarrow{\alpha^*} H^0(X,\mathcal{B}) \xrightarrow{\beta^*} H^0(X,\mathcal{C}) \xrightarrow{\delta^*} H^1(X,\mathcal{A}) \xrightarrow{\alpha^*} H^1(X,\mathcal{B}) \xrightarrow{\beta^*} \dots$$

Proof. By Lemma 2.45 we have a commutative diagram

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & C^p(X,\mathcal{A}) \arrow[r, "\tilde{\alpha}"] \arrow[d, "d"] & C^p(X,\mathcal{B}) \arrow[r, "\tilde{\beta}"] \arrow[d, "d"] & C_0^p(X,\mathcal{C}) \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & C^{p+1}(X,\mathcal{A}) \arrow[r, "\tilde{\alpha}"] \arrow[d] & C^{p+1}(X,\mathcal{B}) \arrow[r, "\tilde{\beta}"] \arrow[d] & C_0^{p+1}(X,\mathcal{C}) \arrow[r] \arrow[d] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

in which the rows are exact and the columns are complexes. The definitions of α^* and β^* are obvious from the commutativity of the diagram, and exactness at $H^p(X,\mathcal{B})$ is trivial. We now construct the homomorphism δ^*.

Let $[s] \in H^p(X,\mathcal{C})$ with representative cocycle s. Invoking Lemma 2.47 we may regard $s \in C_0^p(X,\mathcal{C})$ with $d(s) = 0$. Since $\tilde{\beta}$ is surjective there exists $t \in C^p(X,\mathcal{B})$ with $\tilde{\beta}(t) = s$. Now by commutativity $d(g) \in C^{p+1}(X,\mathcal{B})$ is in $\ker(\tilde{\beta}) = \operatorname{im}(\tilde{\alpha})$. So there exists $f \in C^{p+1}(X,\mathcal{A})$ such that $\tilde{\alpha}(f) = d(g)$. Again by commutativity we have $d(f) = 0$, so $[f] \in H^{p+1}(X,\mathcal{A})$. We define $\delta^*([s]) = [f]$.

To check that this is well-defined is easy diagram chasing. Similarly the exactness of the sequence at $H^p(X,\mathcal{A})$ and $H^p(X,\mathcal{C})$ is elementary yet tedious. Proofs may be found in any homological algebra text. $\square$

Remark 2.49. This theorem underpins the power of cohomological methods. It is ubiquitous in the arguments of §6.

Example 2.50. We reconsider the exponential sheaf sequence defined in Example 2.27. The corresponding long exact sequence of cohomology is given by

$$0 \longrightarrow H^0(X,\mathbb{Z}) \longrightarrow H^0(X,\mathcal{O}) \longrightarrow H^0(X,\mathcal{O}^*) \longrightarrow H^1(X,\mathbb{Z}) \longrightarrow \dots$$

In particular we now know immediately that the short sequence of global sections is exact iff $H^1(X,\mathbb{Z}) = 0$. Thus if g is a nonzero holomorphic function on X, we may define a holomorphic branch of $\log(g)$ on X iff X is simply connected.

3 Bundles

In elementary twistor theory the most important sheaves arise as sections of vector bundles. Moreover, Yang-Mills theory is best described in terms of connections on fibre bundles. Our ultimate goal, a sketch of the Penrose-Ward transform, will rely on both of these ingredients. This serves as good motivation for formally studying the mathematics of bundles.

We begin by rigorously reviewing the basic properties of fibre bundles, including their construction via transition functions. We specialize to the important cases of vector and principal bundles, demonstrating their natural association. Finally we briefly introduce connections on principal bundles, using the resulting covariant derivative on a trivial vector bundle to formulate Yang-Mills theory in mathematically precise language. For brevity we omit several technical proofs without explanation, referring the reader to Nakahara [28].

The mathematically experienced or physically motivated reader might prefer to skip this section, referring to it later in the text as necessary. Nevertheless a thorough study of this material will reward the audience with some useful gems, such as Sparling's contour integral formulae.

3.1 Fibre Bundles

Definition 3.1. A **fibre bundle** over a topological space X is a collection (E, π, F) satisfying the following conditions

1. E and F are topological spaces.
2. $\pi : E \longrightarrow X$ is a continuous surjection.
3. For all $x \in X$ there is a neighbourhood $U \ni x$ and a homeomorphism $\varphi : \pi^{-1}(U) \longrightarrow U \times F$ making the following diagram commute

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"] & U \times F \arrow[dl, "\mathrm{proj}_1"] \\ U & \end{tikzcd}$$

We call E the **total space**, X the **base space**, π the **projection**, F the **fibre**, and $\{(U, \varphi)\}$ a **local trivialisation**.

Remark 3.2. Morally, a fibre bundle is a space E which is locally a direct product of spaces X and F.

Example 3.3. The direct product $X \times F$ is called the **trivial bundle** with fibre F over X.

Definition 3.4. A **local section** of the fibre bundle (E, π, F, X) over an open set $U \subset X$ is a map $s : U \longrightarrow E$ such that $\pi \circ s = \mathrm{id}_X$. The space of local sections over U is denoted $\Gamma(U, E)$.

Remark 3.5. The sections of a fibre bundle form a sheaf on X. We abuse notation by referring to this sheaf as E, when it is convenient.

Definition 3.6. Let (φ_i, U_i) and (φ_j, U_j) be two local trivialisations with $U_{ij} = U_i \cap U_j \neq \emptyset$. Then on $U_{ij} \times F$ we define the **transfer function** $T_{ij} = \varphi_i \circ \varphi_j^{-1}$.

Remark 3.7. This is a homeomorphism by definition of φ_i and φ_j.

Definition 3.8. Denote the homeomorphism group of F by $\mathrm{Homeo}(F)$. Define the **transition function** $t_{ij} : U_{ij} \longrightarrow \mathrm{Homeo}(F)$ by

$$T_{ij}(x, f) = (x, t_{ij}(x)f)$$

Remark 3.9. The transition functions for a fibre bundle tell us how to glue together the locally trivial areas on overlaps. They can be regarded as encoding the twisting of the fibre bundle. Clearly if E is the trivial bundle $X \times F$ then one can choose all transition functions such that $t_{ij}(x) = \mathrm{id}_F$.

Lemma 3.10. The transition functions satisfy the following relations

1. $t_{ii}(x) = \mathrm{id}_F$ on U_i.
2. $t_{ij}(x)t_{ji}(x) = \mathrm{id}_F$ on $U_i \cap U_j$.
3. $t_{ij}(x)t_{jk}(x)t_{ki}(x) = \mathrm{id}_F$ on $U_i \cap U_j \cap U_k$.

Proof. Trivial from the definition. $\square$

Remark 3.11. Apply the language of Čech cohomology to maps $U \longrightarrow \mathrm{Homeo}(F)$ taking the abelian group operation to be pointwise multiplication. The conditions 2 and 3 then say that the transition functions $\{t_{ij}\}$ form a 1-cochain and a 1-cocycle respectively.

Theorem 3.12 (Reconstructing Fibre Bundles)**.** Let X be a space with open covering $\{U_i\}$. Suppose we are given a space F, a group $G \leq \mathrm{Homeo}(F)$ and functions $t_{ij} : U_{ij} \longrightarrow G$ satisfying the 1-cocycle condition. Then there exists a fibre bundle E over X with fibre F and transition functions t_{ij}.

Proof. Let $\tilde{E} = \bigsqcup_i (U_i \times F)$ endowed with the product topology. Define an equivalence relation on $\tilde{E}$ by

$$(x, f) \sim (y, g) \text{ iff } x = y \text{ and } g = t_{ij}(x)f$$

whenever $(x, f) \in U_j \times F$ and $(y, g) \in U_i \times F$. Note that we required the cocycle condition for this to be transitive. Now we let $E = \tilde{E}/\sim$ endowed with the quotient topology.

There is a natural projection $\pi : E \longrightarrow X$ given by $\pi([x, f]) = x$. We define local trivialisations $\varphi_j([x, f]) = (x, f)$, which are homeomorphisms by construction of E, and clearly satisfy the required commutative diagram. Finally on U_{ij} we have $\varphi_i \circ \varphi_j^{-1}(x, f) = (x, t_{ij}(x)f)$ so the transition functions are t_{ij}. $\square$

Remark 3.13. We have an immediate converse to the statement in Remark 3.9, namely if we can choose all transition functions such that $t_{ij}(x) = \mathrm{id}_F$ then the bundle is trivial.

Lemma 3.14. Let (E, π, F) be a fibre bundle over X with transition functions t_{ij} relative to a covering $\{U_i\}$ of X. Suppose we are given a collection of maps $f_i : U_i \longrightarrow F$ satisfying on U_{ij}

$$f_j(x) = t_{ji}(x)f_i(x)$$

Then $\{f_i\}$ determines a global section of E and all global sections arise in this way.

Proof. Let $\varphi_i : \pi^{-1}(U_i) \longrightarrow U_i \times F$ be the local trivialisations inducing the transition functions t_{ij}. Then f_i determines a local section $\tilde{f}_i$ of E over U_i by

$$\tilde{f}_i(x) = \varphi_i^{-1}(x, f_i(x))$$

Now on U_{ij} we have

$$\tilde{f}_j(x) = \varphi_j^{-1}(x, f_j(x)) = \varphi_j^{-1}(x, t_{ji}f_i(x)) = \varphi_j^{-1}\varphi_j\varphi_i^{-1}(x, f_i(x)) = \tilde{f}_i(x)$$

so the local sections glue to form a global section $\tilde{f}$. Conversely if $\tilde{f}$ is a global section then by restriction we obtain local sections $\tilde{f}_i$ on U_i with $\tilde{f}_i = \tilde{f}_j$ on U_{ij}. Defining $f_i(x) = \mathrm{proj}_2 \circ \varphi_i \circ \tilde{f}_i(x)$ we have

$$(x, f_j(x)) = \varphi_j\varphi_i^{-1}(x, f_i(x)) = (x, t_{ji}f_i(x))$$

on U_{ij} as required. $\square$

Definition 3.15. Let (E, π, F) be a fibre bundle over X, and G a subgroup of $\text{Homeo}(F)$. A ***G*-atlas** for (E, π, F) is a collection $\{(U_i, \varphi_i)\}$ of local trivialisations such that $X = \bigcup U_i$ and the induced transition functions are G-valued.

Definition 3.16. A ***G*-bundle** (E, π, F, G) is a fibre bundle with a maximal G-atlas. G is called the **structure group** of the bundle.

Remark 3.17. By definition of transition functions we consider the structure group G to have a natural *left action* on the fibre F. In section 3.3 we see that for a certain class of bundles one can also define a *right action* of G on the total space E. This distinction is conceptually important as we develop the theory.

Lemma 3.18. *Consider a G-bundle (E, π, F) over X. Let H be the set of transition functions at $x \in X$. Then $H = G$.*

Proof. Clearly $H \subset G$. Let $g \in G$ and $h \in H$. Then there are local trivialisations φ_i and φ_j in some neighbourhood U of x such that $(x, h.f) = \varphi_i \circ \varphi_j(x, f)$ for all $f \in F$. Define $\varphi_k = (\text{id}_U \times gh^{-1}) \circ \varphi_i : \pi^{-1}(U) \longrightarrow U \times F$ a local trivialisation. Note that φ_k must be in the G-atlas of E for it is maximal. Moreover $\varphi_k \circ \varphi_j(x, f) = (x, gh^{-1}h.f) = (x, g.f)$ so $g \in H$ as required. $\square$

Remark 3.19. Every fibre bundle can be considered as a G-bundle by choosing $G = \text{Homeo}(F)$. More generally an H-bundle is clearly a G-bundle if $H \leq G$. The converse is more subtle, and motivates the following definition.

Definition 3.20. Let E be a G-bundle, and suppose there exists a choice of local trivialisations such that the transition functions take values in $H \leq G$. Then we say that the structure group of E is **reducible** to H.

Example 3.21. A bundle is trivial iff its structure group is reducible to $\{\text{id}\}$.

Remark 3.22. Following Isham [23] we note without proof that the reducibility of structure groups is related to spontaneous symmetry breaking in Yang-Mills theory and the identification of Riemannian metrics in differential geometry.

Definition 3.23. Let (E_i, π_i, F_i) be fibre bundles over X_i for $i = 1, 2$. A **morphism** of fibre bundles is a continuous map $\tilde{f} : E_1 \longrightarrow E_2$ mapping each fibre $\pi_1^{-1}(x)$ of E_1 onto a fibre $\pi_2^{-1}(y)$ of E_2.

Lemma 3.24. *$\tilde{f}$ induces a continuous map f on base spaces satisfying the commutative diagram*

$$\begin{array}{ccc} E_1 & \xrightarrow{\tilde{f}} & E_2 \\ \big\downarrow{\scriptstyle \pi_1} & & \big\downarrow{\scriptstyle \pi_2} \\ X_1 & \xrightarrow{f} & X_2 \end{array}$$

Proof. The definition of the map is trivial, so we only need prove continuity. This follows from the continuity and openness of π_1 and π_2, viz. Naber [27, p. 67]. $\square$

Definition 3.25. Suppose (E_i, π_i, F_i) are fibre bundles over the same base X for $i = 1, 2$. Then a bundle morphism $\tilde{f} : E_1 \longrightarrow E_2$ is an **equivalence** if $\tilde{f}$ is a homeomorphism and the induced map $f = \mathrm{id}_X$. An **automorphism** of E_1 is an equivalence $\tilde{f} : E_1 \longrightarrow E_1$.

Lemma 3.26. Let E and E' be G-bundles over X with the same fibre. Suppose also that they have common trivialising neighbourhoods $\{U_i\}$. Let g_{ij} and g'_{ij} denote their transition functions. Then E and E' are equivalent iff there exist continuous functions $\lambda_i : U_i \longrightarrow G$ such that

$$g'_{ij}(x) = \lambda_i(x)^{-1} g_{ij}(x) \lambda_j(x)$$

Proof. ($\Rightarrow$) Let $\tilde{\alpha} : E \longrightarrow E'$ be an equivalence. Choose local trivialisations φ_i, φ'_i on $\pi^{-1}(U_i)$. We define $\lambda_i : U_i \longrightarrow G$ by

$$\lambda_i(x).f = \varphi_i \circ \tilde{\alpha}^{-1} \circ \varphi_i'^{-1}(x, f)$$

which is well-defined since $\tilde{\alpha}$ is an equivalence. Now we check

$$\lambda_i(x)^{-1} g_{ij}(x) \lambda_j(x).f = \varphi'_i \circ \tilde{\alpha} \circ \varphi_i^{-1} \circ \varphi_i \circ \varphi_j^{-1} \circ \varphi_j \circ \tilde{\alpha}^{-1} \circ \varphi_j'^{-1}(x, f)$$

whence the required relation holds.

($\Leftarrow$) Define $\tilde{\alpha}$ locally on $\pi^{-1}(U_i)$ as follows. Let $p \in \pi^{-1}(U_i)$ with $\varphi_i(p) = (x, f_i)$. Then take

$$\tilde{\alpha}_i(p) = \varphi_i'^{-1}(x, \lambda_i(x)^{-1}.f_i)$$

Clearly this is a homeomorphism, so it only remains to check compatibility on overlaps $U_i \cap U_j$. We calculate

$$\begin{aligned}
\tilde{\alpha}_j(p) &= \varphi_i'^{-1} \circ \varphi'_i \circ \varphi_j'^{-1}(x, \lambda_j(x)^{-1}.f_j) \\
&= \varphi_i'^{-1}(x, g'_{ij}(x) g'_{ij}(x)^{-1} \lambda_i(x)^{-1} g_{ij}(x) g_{ij}(x)^{-1}.f_i)
\end{aligned}$$

and the result follows. $\square$

Lemma 3.27. If X is contractible then every fibre bundle over X is equivalent to the trivial bundle.

Proof. See Steenrod [36, p. 53]. $\square$

Lemma 3.28. If $f : A \longrightarrow X$ is nullhomotopic then it extends to a map $f : CA \longrightarrow X$ where CA is the cone over A.

Proof. Recall the definition $CA = (A \times I)/(A \times \{1\})$. A nullhomotopy $F : A \times I \longrightarrow X$ is a continuous map with $F(a, 0) = f(a)$ and $F(a, 1) = k$ for some constant $k \in X$. Then F is precisely an extension of f to CA. $\square$

Theorem 3.29. The equivalence class of a G-bundle E over S^n is completely determined by the homotopy class of its transition functions.

Proof. Define an open covering of S^n by taking $U_S = S^n \setminus \{\text{north pole}\}$ and $U_N = S^n \setminus \{\text{south pole}\}$. Since U_N and U_S are contractible, E reduces to a trivial bundle over each of them. The structure of E is hence completely determined by the transition function $t_{NS} : U_N \cap U_S \longrightarrow G$.

Suppose t_{NS} and t'_{NS} are homotopic transition functions, defining bundles E and E' respectively. Then $t_{NS} t'^{-1}_{NS}$ is a nullhomotopic function on $U_N \cap U_S$. Since U_N and U_S are cones over $U_N \cap U_S$ we may extend $t_{NS} t'^{-1}_{NS}$ to $\lambda_N : U_N \longrightarrow G$. Define $\lambda_S : U_S \longrightarrow G$ by $\lambda_S(x) = e$. Then on $U_N \cap U_S$ we have

$$\lambda_N(x)^{-1} t_{NS}(x) \lambda_S(x) = t'_{NS}(x) t_{NS}(x)^{-1} t_{NS}(x) e = t'_{NS}(x)$$

so we are done by Lemma 3.26. $\square$

3.2 Vector Bundles

Definition 3.30. A **vector bundle** is a fibre bundle whose fibre is a finite dimensional vector space, and whose transition functions take values in $GL(k)$ where $k = \dim(V)$. A **morphism** of vector bundles is a fibre bundle morphism which is linear on fibres. A vector bundle whose fibre is one-dimensional is called a **line bundle**.

Theorem 3.31 (Classification of Line Bundles)**.** Let $\mathcal{M}$ be a complex manifold. Then $H^1(\mathcal{M}, \mathcal{O}^*) \cong \{$equivalence classes of holomorphic line bundles on $\mathcal{M}\}$.

Proof. A holomorphic line bundle on $\mathcal{M}$ is completely determined by its transition functions with respect to some cover $\{U_i\}$. These are holomorphic functions $f_{ij} : U_{ij} \longrightarrow GL(1, \mathbb{C}) = \mathbb{C}^*$ subject to the conditions of Lemma 3.10. Following Remark 3.11 we note that these precisely require that $\{f_{ij}\} \in Z^1(\{U_i\}, \mathcal{O}^*)$.

From Lemma 3.26 we see that a holomorphic line bundle is equivalent to the trivial bundle iff $f_{ij} = \lambda_j \lambda_i^{-1}$ on U_{ij} for some $\lambda_i \in \mathcal{O}^*(U_i)$. This is exactly the condition $\{f_{ij}\} \in B^1(\{U_i\}, \mathcal{O}^*)$. In other words, inequivalent holomorphic line bundles with trivialising neighbourhoods $\{U_i\}$ biject with elements of $H^1(\{U_i\}, \mathcal{O}^*)$. Taking the limit of progressively finer covers yields the result. $\square$

Corollary 3.32. The group of equivalence classes of holomorphic line bundles over $\mathbb{P}^1$ is isomorphic to $\mathbb{Z}$.

Proof. Recall the exponential sheaf sequence (Example 2.27) and consider the section

$$H^1(\mathbb{P}^1, \mathcal{O}) \longrightarrow H^1(\mathbb{P}^1, \mathcal{O}^*) \longrightarrow H^2(\mathbb{P}^1, \mathbb{Z}) \longrightarrow H^2(\mathbb{P}^1, \mathcal{O})$$

of the induced cohomology sequence. Now $H^1(\mathbb{P}^1, \mathcal{O}) = 0$ by Example 2.43 and $H^2(\mathbb{P}^1, \mathcal{O}) = 0$ since our Leray cover of $\mathbb{P}^1$ has only 2 open sets. Finally, $H^2(\mathbb{P}^1, \mathbb{Z}) = \mathbb{Z}$ since $\mathbb{P}^1$ is topologically a sphere. The result follows by the above theorem. □

Example 3.33 (Tensor Product Bundles). Let (E_1, π_1, V_1) and (E_2, π_2, V_2) be vector bundles over X with transition functions g_{ij} and h_{ij} relative to some fixed cover $\{U_i\}$ of X. We define the tensor product bundle $E_1 \otimes E_2$ to have fibre $V_1 \otimes V_2$ and transition functions $g_{ij} \otimes h_{ij}$ pointwise. This defines a vector bundle over X by the reconstruction theorem (Theorem 3.12).

Example 3.34 (Dual Bundles). Let (E, π, V) be a vector bundle over X with transition functions g_{ij} relative to the cover $\{U_i\}$. The dual bundle E^* is defined to be the vector bundle with fibre V^* and transition functions $(g_{ij}^{-1})^*$ pointwise, where g^* denotes the transpose of g. These transition functions are chosen so that the cocycle condition in Theorem 3.12 is naturally satisfied.

Example 3.35 (Line Bundles). Let (E, π, V) be a line bundle with transition functions $g_{ij} : U_{ij} \longrightarrow GL(\mathbb{C}) \cong \mathbb{C}$. We identify $V^{\otimes n} \cong V$ by sending $e \otimes \cdots \otimes e \longmapsto e$ for some $0 \neq e \in V$ and extending linearly. We therefore regard $E^{\otimes n}$ as a line bundle with fibre V and transition functions $g_{ij}^n(x) \equiv g_{ij}(x)^n \in \mathbb{C}$.

Similarly we may identify $V \cong V^*$ by sending $e \longmapsto (\lambda \longmapsto e\lambda)$ for all $e \in V$. This allows us to identify g^* with g for all V-automorphisms g. Therefore we regard E^* as a line bundle with fibre V and transition functions $g_{ij}^{-1}(x) \equiv g_{ij}(x)^{-1} \in \mathbb{C}$.

Example 3.36 (Tautological Bundle). Endow $\mathbb{C}^{n+1}$ with coordinates $z = (z_0, \ldots, z_n)$ and $\mathbb{P}^n$ with homogeneous coordinates $[z] = [z_0 : \cdots : z_n]$. We define a line bundle $\mathcal{O}(-1)$ over $\mathbb{P}^n$ by choosing the fibre over $[z]$ to be the line in $\mathbb{C}^{n+1}$ defined by $[z]$. Consider the open cover of $\mathbb{P}^n$ given by $U_i = \{[z] \text{ such that } z_i \neq 0\}$. We define local trivialisations on $\pi^{-1}(U_i) \subset \mathbb{P}^n \times \mathbb{C}^{n+1}$ by

$$\varphi_i([z], \lambda z) = ([z], \lambda z_i)$$

which are easily seen to be homeomorphisms and linear on fibres, where $\lambda \in \mathbb{C}$.

The transition functions satisfy

$$([z], t_{ij}([z])\lambda) = \varphi_i \circ \varphi_j^{-1}([z], \lambda) = \varphi_i([z], (\lambda/z_j)z) = ([z], (z_i/z_j)\lambda)$$

so $t_{ij}([z]) = z_i/z_j$.

We further define $\mathcal{O}(1) = \mathcal{O}(-1)^*$, $\mathcal{O}(k) = \mathcal{O}(1)^{\otimes k}$ and $\mathcal{O}(-k) = \mathcal{O}(-1)^{\otimes k}$ for $k \in \mathbb{N}$. By the preceding examples these are line bundles over $\mathbb{P}^n$ with the same fibre as $\mathcal{O}(-1)$ and transition functions

$$t_{ij}([z]) = \begin{cases} (z_j/z_i)^k & \text{for } \mathcal{O}(k) \\ (z_i/z_j)^k & \text{for } \mathcal{O}(-k) \end{cases}$$

Finally defining $\mathcal{O}$ as $\mathcal{O}(1) \otimes \mathcal{O}(-1)$ we note that $\mathcal{O}$ has transition functions $t_{ij}([z]) = \mathrm{id}_{\mathbb{C}}$, so $\mathcal{O}$ is the trivial bundle over $\mathbb{P}^n$. Thus the sections of $\mathcal{O}$ form the sheaf $\mathcal{O}$ we encountered in Example 2.4.

Lemma 3.37. Let $\mathbb{C}[z^0, \dots z^n]_k$ be the ring of homogeneous polynomials of degree k in $n+1$ variables and Γ denote holomorphic sections. Then

$$\Gamma(\mathbb{P}^n, \mathcal{O}(k)) \cong \begin{cases} \mathbb{C}[z^0, \dots z^n]_k & \text{for } k \geq 0 \\ 0 & \text{for } k < 0 \end{cases}$$

Proof. Use the usual open cover $U_i = \{[z] \text{ such that } z_i \neq 0\}$ for $\mathbb{P}^n$. By Lemma 3.14 a global section of $\mathcal{O}(k)$ is a collection of holomorphic functions $s_i : U_i \cong \mathbb{C}^n \longrightarrow \mathbb{C}$ such that $s_j(x) = t_{ji}(x)s_i(x)$ on U_{ij}. Hence let $\{s_i\} \in \Gamma(\mathbb{P}^n, \mathcal{O}(k))$. Fix overlapping open sets U_i and U_j. Choose local coordinates

$$\begin{aligned} u &= (u^0, \dots \widehat{u^i}, \dots u^n) = (z^0/z^i, \dots \widehat{z^i/z^i}, \dots z^n/z^i) \text{ for } U_i \\ v &= (v^0, \dots \widehat{v^j}, \dots v^n) = (z^0/z^j, \dots \widehat{z^j/z^j}, \dots z^n/z^j) \text{ for } U_j \end{aligned}$$

where the hat denotes omission. Then by Example 3.36 we have the condition

$$s_j(v) = (v^i)^k s_i(u) \qquad (\star)$$

on the intersection U_{ij}. Now expand the functions s_i and s_j as power series in u^j and v^i respectively, writing

$$\begin{aligned} s_i(u) &= \sum_{\alpha=0}^{\infty} a_\alpha (u^j)^\alpha \\ s_j(v) &= \sum_{\alpha=0}^{\infty} b_\alpha (v^i)^\alpha \end{aligned}$$

where $a_\alpha \equiv a_\alpha(u^0, \ldots \widehat{u^i}, \ldots \widehat{u^j}, \ldots u^n)$ and $b_\alpha \equiv b_\alpha(v^0, \ldots \widehat{v^i}, \ldots \widehat{v^j}, \ldots v^n)$.

Noting that $u^j = (v^i)^{-1}$, equation $(\star)$ becomes

$$\sum_{\alpha=0}^{\infty} b_\alpha (v^i)^\alpha = (v^i)^k \sum_{\alpha=0}^{\infty} a_\alpha (v^i)^{-\alpha} \tag{$\dagger$}$$

Equating coefficients we find that $b_\alpha = a_\alpha = 0$ for $\alpha > k$, and $b_\alpha = a_{k-\alpha}$ for $0 \leq \alpha \leq k$. We immediately see that if $k < 0$ then $\Gamma(\mathbb{P}^n, \mathcal{O}(k)) \cong 0$.

Now suppose $k \geq 0$. Equation $(\dagger)$ says that $s_i : U_i \longrightarrow \mathbb{C}$ is a polynomial function in the n variables $\{z_l/z_i : 0 \leq l \leq n,\ i \neq l\}$. In other words

$$s_i \in k[z_0/z_i, \ldots \widehat{z_i/z_i}, \ldots z_n/z_i]$$

The transition condition $(\star)$ then forces

$$(z_i/z_j)^k s_i \in k[z_0/z_j, \ldots \widehat{z_j/z_j}, \ldots z_n/z_j]$$

so certainly

$$(z_i)^k s_i \in k[z_0, \ldots z_n]$$

Moreover $(z_i)^k s_i$ must be a homogeneous polynomial of degree k.

We therefore define a ring homomorphism $\varphi : \Gamma(\mathbb{P}^n, \mathcal{O}(k)) \longrightarrow \mathbb{C}[z^0, \ldots z^n]_k$ by $\{s_i\} \longmapsto f$ where

$$f(z) = s_i([z])(z^i)^k \text{ for } [z] \in U_i$$

This is well-defined by our earlier observations, and the homomorphism property is trivial. Finally the inverse

$$s_i([z]) = f(z)/(z^i)^k$$

is well-defined by reversing the argument above. $\square$

Remark 3.38. This confirms that $H^0(\mathbb{P}^1, \mathcal{O}) = \mathbb{C}$ as seen in Example 2.43. We further note that

$$H^0(\mathbb{P}^1, \mathcal{O}(k)) \cong \begin{cases} \mathbb{C}^{k+1} & \text{for } k \geq 0 \\ 0 & \text{for } k < 0 \end{cases}$$

Remark 3.39. Note that local sections of $\mathcal{O}(-k)$ for $k > 0$ over $U \subset U_i \subset \mathbb{P}^n$ may be regarded either as holomorphic functions on U, or as homogeneous polynomials of degree $-k$ in the $n+1$ variables $(z_0, \ldots z_n)$.

Example 3.40 (Sparling's Formula). Work in homogeneous coordinates $[\pi_{0'} : \pi_{1'}]$

for $\mathbb{P}^1$. Let $f_{01} \in Z^1(\mathbb{P}^1, \mathcal{O}(-1))$ and regard it as a holomorphic function on U_{01}. Define $g_0, g_1 \in C^0(\mathbb{P}^1, \mathcal{O}(-1))$ by

$$g_i(\pi_{A'}) = \frac{1}{2\pi i} \oint_{\Gamma_i} (\xi^{B'} \pi_{B'})^{-1} f_{01}(\xi_{A'}) \xi_{C'} d\xi^{C'}$$

where the Γ_i are taken to be two contours in U_{01} surrounding $\pi_{A'}$ as shown in Figure A. Note that each g_i is a homogeneous polynomial of degree -1 in the $\pi_{A'}$ so genuinely defines a section of $\mathcal{O}(-1)$ over U_i by Remark 3.39.

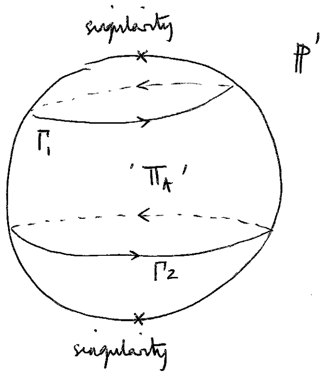

Figure A

We claim that $d\{g_i\} = g_0 - g_1 = f_{01}$ on U_{01}. To prove this we work in the coordinate chart $\xi_{C'} = (\lambda, 1)$ on U_1 so that

$$g_i(\pi_{A'}) = \frac{1}{2\pi i} \oint_{\Gamma_i} \frac{f_{01}(\lambda, 1)}{\pi_{0'} - \lambda \pi_{1'}} d\lambda$$

whence we find that

$$\begin{aligned} g_0(\pi_{A'}) - g_1(\pi_{A'}) &= \frac{1}{2\pi i} \oint_{\Gamma} \frac{f_{01}(\lambda, 1)}{\pi_{0'} - \lambda \pi_{1'}} d\lambda \\ &= f_{01}\left(\pi_{0'}/\pi_{1'}, 1\right) \\ &= f_{01}(\pi_{0'}, \pi_{1'}) \end{aligned}$$

where the contour Γ in $\mathbb{C}$ is as illustrated in Figure B. We therefore conclude that every cocycle in $C^1(\mathbb{P}^1, \mathcal{O}(-1)$ is a coboundary, so $H^1(\mathbb{P}^1, \mathcal{O}(-1)) = 0$.

Remark 3.41. The notation employed above is more fully explained in §4. Observe in particular the following subtlety. We regard $g_i(\pi_{A'})$ as a homogeneous polynomial of degree -1 in the two variables $(\pi_{0'}, \pi_{1'})$, whereas we view $f_{01}(\pi_{A'})$ as a holomorphic function of the homogeneous coordinates $[\pi_{0'} : \pi_{1'}]$. Clearly our notation $g_i(\pi_{A'})$ and $f_{01}(\pi_{A'})$ does not clearly distinguish these.

Using Remark 3.39 and the proof of Lemma 3.37 we could easily rewrite

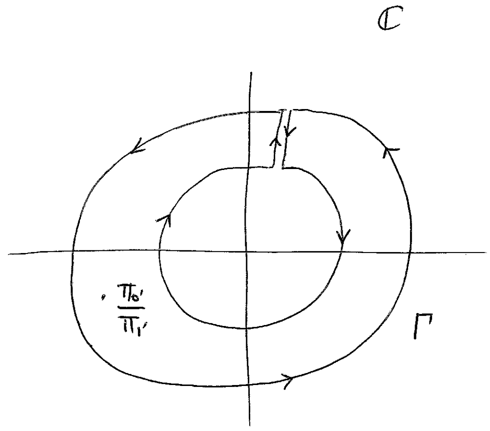

Figure B

Sparling's formula using only one of these possible viewpoints. Although superficially clearer, this is more cumbersome in the long run. Henceforth we shall rely on the reader to determine from which perspective we are examining local sections of $\mathcal{O}(k)$. The answer should always be clear from the context.

Theorem 3.42.

$$H^1(\mathbb{P}^1, \mathcal{O}(k)) \cong \begin{cases} 0 & \text{for } k \geq -1 \\ \mathbb{C}^{-k-1} & \text{for } k < -1 \end{cases}$$

Proof. We treat first the case the $k \geq -1$. Let $f_{01} \in Z^1(\mathbb{P}^1, \mathcal{O}(k))$ and view $f_{01}(\pi_{A'})$ as a homogeneous polynomial of degree k. Let $\alpha(\pi_{A'})$ be a homogeneous polynomial of degree $(k+1)$. Then $f_{01}(\pi_{A'})/\alpha(\pi_{A'}) \in Z^1(\mathbb{P}^1, \mathcal{O}(-1))$ so we may split it using Sparling's formula as

$$f_{01}(\pi_{A'})/\alpha(\pi_{A'}) = g_0(\pi_{A'}) - g_1(\pi_{A'})$$

with $g_0, g_1 \in C^0(\mathbb{P}^1, \mathcal{O}(-1))$. Multiplying through by $\alpha(\pi_{A'})$ we immediately exhibit f_{01} as a coboundary. Hence $H^1(\mathbb{P}^1, \mathcal{O}(k)) = 0$.

Now suppose $k < -1$. Let $f_{01} \in Z^1(\mathbb{P}^1, \mathcal{O}(k))$ and view $f_{01}(\pi_{A'})$ as a homogeneous polynomial of degree k. Now we may calculate

$$\begin{aligned} f_{01}(\pi_{0'}, \pi_{1'}) &= (\pi_{1'})^k f_{01}(\pi_{0'}/\pi_{1'}, 1) \\ &= (\pi_{1'})^k \sum_{r=-\infty}^{\infty} a_r \left(\frac{\pi_{0'}}{\pi_{1'}}\right)^r \\ &= \sum_{r=-k}^{\infty} a_{-r} \frac{(\pi_{1'})^{k+r}}{(\pi_{0'})^r} + \sum_{r=0}^{\infty} a_r \frac{(\pi_{0'})^r}{(\pi_{1'})^{r-k}} + \sum_{r=1}^{-k-1} a_{-r} \frac{(\pi_{1'})^{k+r}}{(\pi_{0'})^r} \end{aligned}$$

The first two terms are well-defined on U_0 and U_1 respectively. Moreover they are both homogeneous of degree k in $(\pi_{0'}, \pi_{1'})$. Therefore together they define an element of $C^0(\mathbb{P}^1, \mathcal{O}(k))$. In other words, f_{01} is cohomologous to

$$\tilde{f}_{01}(\pi_{A'}) = \sum_{r=1}^{-k-1} a_{-r} \frac{(\pi_{1'})^{k+r}}{(\pi_{0'})^r}$$

Observe that this is well-defined only on U_{01}, so we cannot simplify futher. $\tilde{f}_{01}$ has $-k-1$ free complex parameters, namely the a_r. Therefore we conclude that $H^1(\mathbb{P}^1, \mathcal{O}(k)) = \mathbb{C}^{-k-1}$. □

Remark 3.43. The evident isomorphism $H^1(\mathbb{P}^1, \mathcal{O}(-k-2)) \cong H^0(\mathbb{P}^1, \mathcal{O}(k))$ is no coincidence. It is a special case of **Serre duality**, which is treated fully in Hartshorne [17, §III.7].

3.3 Principal Bundles

Definition 3.44. Let G be a topological group. A **principal G-bundle** is a G-bundle with fibre G, where the transition functions act on G by left multiplication. The structure group G is called the **gauge group**.

Lemma 3.45. *Let (E, π, G) be a principal bundle over X. Then there a continuous right action σ of G on E defined fibrewise by right multiplication.*

Proof. We must check that σ is well-defined on E. Consider local trivialisations $\varphi_i : \pi^{-1}(U_i) \longrightarrow U_i \times G$. Let $x \in U_i \cap U_j$ and $u \in \pi^{-1}(x)$. Define $\varphi_i(u) = (x, f_i)$ and $\varphi_j(u) = (x, f_j)$ so $f_j = t_{ji}(x) f_i$. Write σ_i for $\sigma|_{U_i}$. Then we have

$$\sigma_i(u, g) \equiv \varphi_i^{-1}(x, f_i g) = \varphi_j^{-1}(x, t_{ji}(x) f_i g) = \varphi_j^{-1}(x, f_j g) \equiv \sigma_j(u, g)$$

so σ is independent of choice of local trivialisation. Continuity follows from the continuity of the trivialisations and group operation. □

Remark 3.46. Clearly σ acts freely and transitively on fibres, these being trivial properties of right multiplication. We sometimes write $\sigma(u, g) \equiv u.g$.

Definition 3.47. A **morphism** of principal G-bundles is a fibre bundle morphism $\varphi : E_1 \longrightarrow E_2$ which is G-equivariant with respect to the right action. That is, $\varphi(u.g) = \varphi(u).g$ for all $u \in E_1$ and $g \in G$.

Example 3.48 (Hopf Bundles). We exhibit S^{2n-1} as a principal $U(1)$-bundle over $\mathbb{P}^{n-1}$. We can view S^{2n-1} as a complex $n-1$ sphere, that is

$$S^{2n-1} = \{(z^0, \dots z^n) \in \mathbb{C}^n : |z^0|^2 + \dots + |z^n|^2 = 1\}$$

Define a projection $\pi : S^{2n-1} \longrightarrow \mathbb{P}^{n-1}$ by

$$\pi(z_0, \ldots z_n) = [z_0 : \cdots : z_n]$$

The fibre of π over $[z_0 : \cdots : z_n]$ is given by

$$\{\lambda(z_0, \ldots z_n) : |\lambda| = 1\}$$

which is topologically a circle $U(1)$. We give S^{2n-1} a bundle stucture by choosing trivialising neighbourhoods $U_S = S^{2n-1} \setminus \{\text{north pole}\}$ and $U_N = S^{2n-1} \setminus \{\text{south pole}\}$ and specifying a transition function $t_{NS} : U_N \cap U_S \longrightarrow U(1)$.

Remark 3.49. Suppose $n = 2$, then we find that S^3 is a $U(1)$-bundle over S^2. In this case $U_N \cap U_S$ is homotopy equivalent to $U(1)$. Invoking Theorem 3.29 we see that the bundle structure of S^3 is completely determined by the homotopy class of $t_{NS} : S^1 \longrightarrow S^1$. Therefore equivalence classes of $U(1)$-bundles over S^2 are classified by the fundamental group $\pi^1(U(1)) = \mathbb{Z}$. In particular we may always choose a transition function of the form $t_{NS}(\varphi) = e^{in\varphi}$ with $n \in \mathbb{Z}$.

Theorem 3.50. Let (E, π, G) be a principal bundle over X and $U \subset X$ open. There is a bijection between local sections of E over U and local trivialisations $\varphi : \pi^{-1}(U) \longrightarrow U \times G$. The section associated with a given trivialisation is called a **canonical section**, and vice versa.

Proof. Let $s : U \longrightarrow P$ be a local section. Fix $x \in U$. For each $p \in \pi^{-1}(U)$ there exists a unique $g \in G$ such that $p = s(x).g$, since the right action of G is transitive and free. Define the canonical local trivialisation $\varphi : \pi^{-1}(U) \longrightarrow U \times G$ by $\varphi(u) = (p, g)$. By construction this is a bijection and continuity follows easily, viz. Naber [27, p. 221]. Conversely let $\varphi : \pi^{-1}(U) \longrightarrow U \times G$ be a local trivialisation. Define the canonical section $s : U \longrightarrow P$ by $s(x) = \varphi^{-1}(x, e)$. □

Remark 3.51. In the canonical local trivialisation, the section $s : U \longrightarrow P$ may be viewed as the constant map $s : U \longrightarrow G$ given by $s(x) = e$. In this sense the association is manifestly canonical.

Corollary 3.52. A principal bundle is trivial iff it admits a global section.

Definition 3.53. Let E be a principal G-bundle over X and $U \subset X$ open. A **local gauge** on U is a choice of local section of E over U. Let s and t be local gauges. A **local gauge transformation** from s to t is a smooth map $f : U \longrightarrow G$ such that $t(x) = s(x).f(x)$.

Remark 3.54. Owing to the correspondence in the previous theorem, a local gauge may be thought of as a distinguished local trivialisation for the bundle.

Theorem 3.55. Let (E, π, G) be a principal bundle over X and $U \subset X$ open. There is a bijection between local gauge transformations over U and bundle automorphisms of $\pi^{-1}(U)$.

Proof. Let s be a local section over U and $f : U \longrightarrow G$ a local gauge transformation. Define $\Phi : \pi^{-1}(U) \longrightarrow \pi^{-1}(U)$ by $\Phi(s(x).h) = (s(x).g(x)h)$ for all $h \in G$. Observe that this is a bundle automorphism. Conversely given Φ and s define $t(x) = \Phi^{-1}(s(x))$. Then t is another section of P so there exists $g(x) \in G$ with $t(x) = s(x).g(x)$. Note that g is smooth, and we are done. □

Remark 3.56. We therefore refer to bundle automorphisms of $\pi^{-1}(U)$ as gauge transformations, appealing to this correspondence. This makes the following definition sensible.

Definition 3.57. Let E be a principal G-bundle over X. A **(global) gauge transformation** is a bundle automorphism, i.e. a diffeomorphism $f : E \longrightarrow E$ preserving fibres and commuting with the right action of G on E. The collection of all gauge transformations forms the **group of gauge transformations** denoted $\mathcal{G}(E)$.

Definition 3.58. Let P be a principal G-bundle over X with transition functions t_{ij}. Let ρ be a representation of G on a vector space V. We define the **associated vector bundle** $P \times_\rho V$ over X to have fibre V and transition functions $\rho(t_{ij})$.

Remark 3.59. Suppose ρ is faithful. Then clearly P and $P \times_\rho V$ share the same transition function data. The difference arises in the way the structure group acts on a typical fibre. In particular note that P is trivial iff $P \times_\rho V$ is trivial.

Remark 3.60. In gauge field theories, matter fields interacting with the gauge field are viewed as sections of an associated vector bundle. See Naber [27, §6.8].

Example 3.61 (Adjoint Bundle). Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then we define the adjoint representation Ad of G on $\mathfrak{g}$ by $Ad(g)X = gXg^{-1}$. If P is a principal G-bundle then the associated vector bundle $P \times_{Ad} \mathfrak{g}$ is called the adjoint bundle, and denoted $Ad(P)$.

Remark 3.62. We see in §3.4 that Yang-Mills fields may be regarded as sections of the adjoint bundle associated with a gauge principal bundle.

Example 3.63 (Spin Bundle). Let P be an orthonormal frame bundle over an n-dimensional manifold $\mathcal{M}$. That is to say, P is a principal $SO(n)$-bundle. Recall that $Spin(n)$ is a double cover of $SO(n)$ via some homomorphism φ.

Suppose that P has transition functions t_{ij}. Assume we may lift P to a $Spin(n)$ bundle $\tilde{P}$ over $\mathcal{M}$, in the sense that $\tilde{P}$ has transition functions $\tilde{t}_{ij}$ with

$\varphi(\tilde{t}_{ij}) = t_{ij}$. Let ρ be a representation of $Spin(n)$ on a vector space V which does not descend to a representation of $SO(n)$. Such a representation is called a **spin representation**. The associated bundle $\tilde{P} \times_\rho V$ is called a spin bundle over $\mathcal{M}$.

In the case that $\mathcal{M}$ is a 4-dimensional Lorentzian manifold, we recall that $Spin(1,3) \cong SL(2,\mathbb{C})$. The Weyl spin bundle is then the vector bundle associated to $\tilde{P}$ via the fundamental representation of $SL(2,\mathbb{C})$ on $\mathbb{C}^2$.

Remark 3.64. The lifting of P to $\tilde{P}$ is not always possible. In fact the obstruction is measured by the second Steifel-Whitney class of M, namely $H^2(\mathcal{M}, \mathbb{Z}_2)$. See, for example, Nakahara [28, p. 451].

Definition 3.65. Let E be a vector G-bundle over X with transition functions t_{ij}. The **associated principal bundle** $P(E)$ has fibre G and transition functions t_{ij} acting on G by left multiplication.

Remark 3.66. Note that E is the vector bundle associated to $P(E)$ via the fundamental representation of G.

Example 3.67 (Frame Bundle). Consider the tangent bundle $T\mathcal{M}$ over an n-dimensional manifold $\mathcal{M}$. The associated principal bundle is

$$F\mathcal{M} = \bigsqcup_{p \in \mathcal{M}} F_p\mathcal{M}$$

where $F_p\mathcal{M}$ is the set of frames at p.

Example 3.68. The Hopf bundles S^3 over S^2 are associated principal bundles for the vector bundles $\mathcal{O}(k)$ over $\mathbb{P}^1$. Consider the usual open cover U_i for $\mathbb{P}^1$ with homogeneous coordinates $[z_0 : z_1]$. In Example 3.36 we found transition functions

$$t_{01}([z]) = \left(\frac{z^1}{z^0}\right)^k$$

for $\mathcal{O}(k)$. We may reduce the structure group to $U(1)$ by taking

$$t_{01}([z]) = \left(\frac{z^1|z^0|}{z^0|z^1|}\right)^k$$

Now suppose $[z]$ lies on the equator of $\mathbb{P}^1$. Write $z^j = |z^j|e^{i\varphi^j}$ for $j = 0, 1$ and let $\varphi = \varphi_1 - \varphi_0$ be an equatorial coordinate. Then we obtain transition functions $t_{01}(\varphi) = e^{ik\varphi}$. By Remark 3.49 these define the Hopf bundles over S^2.

3.4 Gauge Fields

Definition 3.69. Let $\mathcal{M}$ be a manifold and V a vector space. Let E be the trivial bundle $\mathcal{M} \times V$. A **V-valued r-form** on M is a section of the bundle

$$\bigwedge^r (T^*\mathcal{M}) \otimes E$$

We denote the $\Gamma(\mathcal{M})$ module of V-valued r-forms by $\Omega^r(\mathcal{M}, V)$.

Remark 3.70. Equivalently a vector-valued form is a smoothly varying collection of linear maps $\omega_p : \bigwedge^r(T_p^*\mathcal{M}) \longrightarrow V$ defined for each $p \in \mathcal{M}$. Clearly an ordinary differential form is merely an $\mathbb{R}$-valued form.

Definition 3.71. Let ω be a V-valued r-form, and choose a basis e_α for V. Then we may write $\omega = \omega^\alpha e_\alpha$, where ω^α are $\mathbb{R}$-valued r-forms. The **exterior derivative** of ω is defined by $d\omega = (d\omega^\alpha)e_\alpha$.

Definition 3.72. Let $\mathfrak{g}$ be a Lie algebra. Suppose $\omega \in \Omega^r(\mathcal{M}, \mathfrak{g})$ and $\eta \in \Omega^s(\mathcal{M}, \mathfrak{g})$. Then the **Lie wedge product** $[\omega \wedge \eta] \in \Omega^{r+s}(\mathcal{M}, \mathfrak{g})$ is defined by

$$[\omega\wedge\eta](X_1, \dots X_{r+s}) = \sum_{\sigma \in S_{r+s}} \operatorname{sgn}(\sigma)[\omega(X_{\sigma(1)}, \dots X_{\sigma(r)}), \eta(X_{\sigma(r+1)}, \dots X_{\sigma(r+s)})]$$

where $[a, b]$ denotes the Lie bracket of $\mathfrak{g}$.

Definition 3.73. Let P be a principal G-bundle over $\mathcal{M}$. Suppose $p \in P$ and $m = \pi(p)$. The **vertical subspace** $V_pP \subset T_pP$ is defined by

$$V_pP = \ker\{\pi_* : T_pP \longrightarrow T_m\mathcal{M}\}$$

Remark 3.74. In other words the vertical subspace consists of those vectors which are tangent to the fibre at p.

Definition 3.75. Define a map $\zeta : \mathfrak{g} \longrightarrow \Gamma(T\mathcal{M})$ by

$$\zeta(A)_p f = \left.\frac{d}{dt}\right|_{t=0} f(p \odot e^{tA})$$

where $\odot$ denotes the right action of G on P. We denote $\zeta(A)$ by $A^\sharp$ and call it the **fundamental vector field** generated by A.

Remark 3.76. It is easy to verify that ζ induces a vector space isomorphism $\mathfrak{g} \longrightarrow V_pP$ as we might expect.

Definition 3.77. A **horizontal subspace** $H_pP \subset T_pP$ is defined to satisfy

$$T_pP = H_pP \oplus V_pP$$

Definition 3.78. A **connection** on a principal G-bundle P is a smoothly varying choice of horizontal subspace H_pP for all $p \in P$ satisfying

$$(R_g)_* H_pP = H_{p\odot g}P$$

Remark 3.79. In a local trivialisation there is evidently a trivial choice of connection, namely $H_pP = T_m\mathcal{M}$ where $m = \pi(p)$. Globally, however, the twisting of the bundle ensures that there is no natural complement to V_pP in T_pP.

Theorem 3.80. A connection on a principal G-bundle P is equivalently a $\mathfrak{g}$-valued 1-form ω satisfying

1. $\omega_p(A^\sharp_p) = A$ for all $A \in \mathfrak{g}$ and $p \in P$.
2. $R_g^*\omega_{p\odot g} = Ad(g^{-1})\omega_p$ for all $g \in G$ and $p \in P$.

Proof. Given $HP \subset TP$ we define ω by

$$\omega_p(X) = \begin{cases} A & \text{if } X = A^\sharp \\ 0 & \text{if } X \in H_pP \end{cases}$$

and extending linearly. Conversely given ω we define HP by

$$H_pP = \{X \in T_pP : \omega(X) = 0\}$$

Algebraic manipulations ensure that the required identities hold. □

Definition 3.81. Let P be a principal G-bundle. Suppose $U \subset \mathcal{M}$ is a trivialising neighbourhood for P. We define a **gauge potential** to be a $\mathfrak{g}$-valued 1-form on U.

Remark 3.82. Henceforth we assume that the structure group G is a matrix group, allowing us to simplify formulae greatly.

Theorem 3.83. Let P be a principal G-bundle over $\mathcal{M}$ with trivialising neighbourhoods U_i and transition functions g_{ij}. A connection on P is equivalently a collection $\{A_i\}$ of gauge potentials satisfying on $U_i \cap U_j$

$$A_j = g_{ij}^{-1}A_ig_{ij} + g_{ij}^{-1}dg_{ij}$$

Proof. See Nakahara [28, p. 377]. □

Remark 3.84. In the case that $\mathcal{M}$ is contractible, P is equivalent to a trivial bundle. Therefore we may take the cover to be $U_1 = \mathcal{M}$, and the overlap condition becomes vacuous. This will often be the case for our purposes, when working over Minkowski space M.

Definition 3.85. A gauge transformation $\Phi : P \longrightarrow P$ acts on a connection HP to yield a connection $H^\Phi P \equiv \Phi_*(HP)$. We say that HP **transforms** to $H^\Phi P$.

Remark 3.86. The fact that $\Phi_*(HP)$ defines a connection is immediate since Φ preserves fibres and is G-equivariant.

Lemma 3.87. Let $\varphi : U \longrightarrow G$ be a local gauge transformation of P. Under φ a gauge potential A on U transforms to

$$A^\varphi = \varphi^{-1} A \varphi + \varphi^{-1} d\varphi$$

Proof. A local gauge transformation amounts to a local change of trivialisation. Therefore the transformation law is given by the formula in Theorem 3.83. □

Remark 3.88. To correctly formalise the correct notion for global gauge transformations requires a little more work. See Figueroa-O'Farrill [13, p. 11] for details.

Definition 3.89. The curvature of a connection ω on P is the $\mathfrak{g}$-valued 2-form on P defined by

$$\Omega = d\omega + [\omega \wedge \omega]$$

Remark 3.90. Geometrically one can show that ω provides a notion of horizontal lifting of curves in $\mathcal{M}$ to P. The curvature then measures the failure of the horizontal lift of a loop to close.

Definition 3.91. The **gauge field** associated to a gauge potential A is the $\mathfrak{g}$-valued 2-form on $\mathcal{M}$ defined by

$$F = dA + [A \wedge A]$$

Remark 3.92. Note immediately that the gauge fields $\{F_i\}$ associated to a connection $\{A_i\}$ precisely encode the curvature of the connection.

Lemma 3.93. The gauge fields $\{F_i\}$ associated to the gauge connection $\{A_i\}$ satisfy on $U_i \cap U_j$

$$F_j = g_{ij}^{-1} F_i g_{ij}$$

Proof. Trivial algebra. □

Remark 3.94. Hence under a local gauge transformation φ we see that F must transform to $F^\varphi = \varphi^{-1} F \varphi$.

Definition 3.95. Let P be a principal G bundle over $\mathcal{M}$ with trivialising neighbourhoods U_i and a connection $\{A_i\}$. Suppose $E = P \times_\rho V$ is an associated vector bundle. Fix a basis e^α for V. Define local sections s^α of E on U_i by $s^\alpha(x) = (x, e^\alpha)$ in the trivialisation induced from P. Define the **associated covariant derivative**

$$D : T\mathcal{M} \times \Gamma(E) \longrightarrow \Gamma(E)$$

in coordinates x^μ on some subset of U_i by taking

$$\begin{aligned} D_\mu s^\alpha &= (\rho_* A_{i\mu})^\beta_\alpha e_\beta \\ D_\mu f &= \partial_\mu f \end{aligned}$$

where $\rho_* : \mathfrak{g} \longrightarrow GL(V)$ is the induced representation of $\mathfrak{g}$ on V, and f is a scalar field on $\mathcal{M}$. For a general vector field $\varphi = \varphi_\alpha s^\alpha$ we therefore have

$$D_\mu \varphi = \partial_\mu \varphi + (\rho_* A_{i\mu})^\beta_\alpha \varphi_\beta$$

Remark 3.96. Note that the transformation law for the A_i precisely means that $D_X\varphi$ defines a global section of E as required.

Example 3.97. Suppose G is a matrix Lie group. If ρ is the fundamental representation then the covariant derivative takes the form

$$D_\mu \varphi = \partial_\mu \varphi + A_{i\mu}\varphi$$

where matrix multiplication is implicit in the final term. If ρ is the adjoint representation then the covariant derivative takes the form

$$D_\mu \varphi = \partial_\mu \varphi + [A_{i\mu}, \varphi]$$

where $[A, B]$ denotes the Lie bracket on $\mathfrak{g}$.

Lemma 3.98 (Bianchi Identity)**.** Fix a principal G-bundle P over $\mathcal{M}$. Consider the curvature F of a gauge potential A on $\mathcal{M}$, viewed as an element of the adjoint bundle $Ad(P)$. In any coordinate chart on $\mathcal{M}$ we have $D_{[\mu}F_{\nu\rho]} = 0$.

Proof. Defining $D_{[\mu}F_{\nu\rho]} = (DF)_{\mu\nu\rho}$ we calculate

$$\begin{aligned} DF = dF + [A \wedge F] = d^2A + [dA \wedge A] - [A \wedge dA] + [A \wedge dA] \\ + [A \wedge A \wedge A] - [dA \wedge F] - [A \wedge A \wedge A] = 0 \end{aligned}$$

as required. □

Definition 3.99. Let P be a principal G-bundle over Minkowski space M, where G is a semisimple matrix group. Since M is contractible, P is trivial so we may work in a single trivialisation. Let A_a denote an arbitrary connection on P and F_{ab} its curvature, written in abstract index notation. The **Yang-Mills action** on M is the functional of A_a given by

$$S[A_a] = -\frac{1}{2}\int_M Tr(F_{ab}F^{ab})$$

Remark 3.100. The condition of semisimplicity ensures that the quadratic form induced from the Killing form on $\mathfrak{g}$ is non-degenerate.

Lemma 3.101. Varying the Yang-Mills action with respect to A and requiring $\delta S = 0$ gives the **Yang-Mills equations**

$$D * F = 0$$

These are a system of nonlinear partial differential equations for A.

Proof. This is trivial but tedious, so we omit it. □

Lemma 3.102. Suppose F is an anti-self-dual gauge field, cf. Definition 4.27. Then F automatically satisfies the Yang-Mills equations.

Proof. Immediate from the Bianchi identity. □

Remark 3.103. This provides an important class of solutions to the Yang-Mills equations. The ASD equation $*F = -iF$ is particularly susceptible to attack by twistor methods. We shall investigate this further in §6.3.

4 Spinor Notation

The machinery of twistor theory is best presented in terms of spinors. These can be regarded as the square root of Minkowskian geometry. Indeed they lie in the fundamental representation of $SL(2,\mathbb{C})$, a double cover of the proper orthochronous Lorentz group. Much as the introduction of the imaginary unit i simplifies and clarifies elementary algebra, the language of spinors allows a unified treatment of physical theories.

We begin by demonstrating the fundamental isomorphism identifying Hermitian spinors with real vectors. This immediately extends to a dictionary between real tensors and higher valence spinors, which we use liberally. Simple algebraic properties of spinors are developed rigorously, including the definition of a covariant derivative on a spinor field.

We rewrite physical field equations in spinor language, to faciliate their solution by twistor methods in §6. There follows a detailed consideration of conformal invariance, a vital part of twistor philosophy. Finally we introduce twistors in a physically motivated manner.

4.1 The Spinor Isomorphism

Definition 4.1. We define **spinor space** to be a 2-dimensional complex vector space S with elements α^A where $A = 0, 1$. These are called **spinors**. $SL(2,\mathbb{C})$ acts on S in the natural way

$$\begin{aligned} \varphi : SL(2,\mathbb{C}) \times S &\longrightarrow S \\ (A, \alpha) &\longmapsto A\alpha \end{aligned}$$

Definition 4.2. We define **conjugate spinor space** to be the 2-dimensional complex vector space S' consisting of the complex conjugates of elements of S. The elements are also called **spinors** but are written $\beta^{A'}$ to distinguish them from elements of S. $SL(2,\mathbb{C})$ acts on S' according to

$$\begin{aligned} \psi : SL(2,\mathbb{C}) \times S' &\longrightarrow S' \\ (A, \beta) &\longmapsto \overline{A}\alpha \end{aligned}$$

Definition 4.3. An element of the tensor product space

$$S \otimes \ldots \otimes S \otimes S^* \otimes \ldots \otimes S^* \otimes S' \otimes \ldots \otimes S' \otimes S'^* \otimes \ldots \otimes S'^*$$

is called a **spinor of higher valence** and denoted $\alpha^{A\ldots BC'\ldots D'}_{E\ldots FG'\ldots H'}$.

Definition 4.4. We define the **conjugation map** $S \longrightarrow S'$ by

$$\alpha^A \longmapsto \overline{\alpha}^{A'} \equiv \overline{\alpha^A}$$

and extend it componentwise to higher valence spinors.

Remark 4.5. Conjugation defines an isomorphism $S \otimes S' \longleftrightarrow S' \otimes S$. In view of this we adopt the convention that the relative ordering of primed and unprimed indices among the upper or among the lower indices is unimportant.

Remark 4.6. Let $\alpha \in S$ and $A \in SL(2, \mathbb{C})$. Observe that $\psi(A, \overline{\alpha}) = \overline{\varphi(A, \alpha)}$.

Definition 4.7. A spinor with equal numbers of primed and unprimed indices is **Hermitian** if $\overline{\alpha}_{A' \dots B' C \dots D} = \alpha_{A' \dots B' C \dots D}$.

Example 4.8. A Hermitian spinor $\alpha_{A'B'CD}$ satifies, for example $\overline{\alpha_{011'1'}} = \alpha_{0'1'11}$.

Lemma 4.9. A 2-index Hermitian spinor is a Hermitian matrix.

Proof. $(\alpha_{AA'})^\dagger = \overline{\alpha_{A'A}} = \overline{\alpha}_{AA'} = \alpha_{AA'}$. □

Theorem 4.10. Fix an arbitrary point $x \in M$. There is a linear isomorphism between real tensors of valence n at x and Hermitian spinors with n primed and n unprimed indices.

Proof. We start with the case $n = 1$. Let V be the vector space of valence 1 tensors. Fix bases for V and S. Define a linear map

$$\Psi(V^a) = V^{AA'} = \frac{1}{\sqrt{2}} \begin{pmatrix} V^0 + V^3 & V^1 + iV^2 \\ V^1 - iV^2 & V^0 - V^3 \end{pmatrix}$$

By the lemma this has the correct codomain, and it is clearly invertible. Now let $n = 2$. Write

$$T^{ab} = \sum T^{\mu\nu} e_\mu \otimes e_\nu$$

where $\{e_\mu\}$ are basis vectors for V. We then define

$$\Psi(T^{ab}) = T^{AA'BB'} = \sum T^{\mu\nu} \Psi(e_\mu) \otimes \Psi(e_\nu)$$

Linearity and invertibility follow from the $n = 1$ case. Continue inductively for higher valence tensors. □

Remark 4.11. Henceforth we shall abuse notation by writing $V^a = V^{AA'}$ and so on. We shall always assume wlog that some bases for V and S have been chosen to effect this identification.

Remark 4.12. Note that we may write $\Psi(V^a) = \sum \sigma^b V^b$ where σ^b is the usual 4-vector of Pauli matrices.

Corollary 4.13. There is a linear isomorphism between complexified tensors of valence n at a point of $\mathbb{C}M$ and spinors with n primed and n unprimed indices.

Definition 4.14. Since S is 2-dimensional there exists a unique non-zero skew 2-index spinor, which we write ϵ_{AB}. Given a basis for S we choose ϵ_{AB} such that $\epsilon_{01} = 1$. The complex conjugate is then $\overline{\epsilon}_{A'B'} = \epsilon_{AB}$ which we shall write $\epsilon_{A'B'}$ for brevity. We similarly define ϵ^{AB} such that $\epsilon^{01} = 1$ and take $\epsilon^{A'B'} = \epsilon^{AB}$. Note that $e^{AB}\epsilon_{CB} = \delta^A_C$.

Lemma 4.15. ϵ_{AB} yields an isomorphism $S \longleftrightarrow S^*$ by

$$\alpha^A = \epsilon^{AB}\alpha_B \text{ and } \alpha_A = \alpha^B\epsilon_{BA}$$

Proof. We check that they are mutually inverse, computing

$$\alpha_A = \epsilon^{BC}\alpha_C\epsilon_{BA} = \delta^C_A\alpha_C = \alpha_A$$

as required. $\square$

Remark 4.16. Care is needed when raising and lowering indices with ϵ since it is skew. A useful mnemonic for our convention is "adjacent indices, descending to the right".

Lemma 4.17. $\alpha^A\beta_A = 0$ iff $\alpha^A = k\beta^A$ some $k \in \mathbb{C}$.

Proof. The backward direction is trivial from the skew-symmetry of ϵ. Conversely $\alpha^A\beta^B\epsilon_{BA} = 0 \Rightarrow \alpha^1\beta^0 = \alpha^0\beta^1 = 0$. Let $k = \alpha^1/\beta^1 = \alpha^0/\beta^0$. $\square$

Definition 4.18. A **(normalised) dyad** for S is a pair of spinors $\{o^A, \iota^A\}$ such that $o^A\iota_A = 1$. By the above lemma $\{o^A, \iota^A\}$ span S.

Lemma 4.19. $||V^a||^2 = 2\det(V^{AA'})$, where $||\cdot||$ denotes the Minkowski norm.

Proof. Trivial from the definition of Ψ. $\square$

Definition 4.20. We say a vector $V^a \in V$ is **future pointing** if $V^0 > 0$. Observe that this concept is invariant under the action of the proper orthochronous Lorentz group on V.

Theorem 4.21. Let $0 \neq V^a \in V$. There exists $0 \neq \alpha^A \in S$ such that $V^{AA'} = \alpha^A\overline{\alpha}^{A'}$ iff V^a is null and future pointing.

Proof. Using the previous lemma we have

$$V^a \text{ null } \Leftrightarrow \det(V^{AA'}) = 0 \Leftrightarrow V^{AA'} \text{ has rank } 1 \Leftrightarrow V^{AA'} \text{ is simple tensor}$$

Therefore V^a is null iff $V^{AA'} = \alpha^A \beta^{A'}$ with $\alpha^A \beta^{A'} = \overline{\alpha}^{A'} \overline{\beta}^A$. Now let $\{\alpha^A, o^A\}$ be a dyad so that $\beta^{A'} = \overline{\alpha}^{A'} \overline{\beta}^A o_A$. Then we find that $\beta^{A'} \overline{\alpha}_{A'} = 0$ whence $\beta^{A'} = k\overline{\alpha}^{A'}$. So V^a is null iff $V^{AA'} = k\alpha^A \overline{\alpha}^{A'}$. Now observe that

$$V^0 = \frac{1}{\sqrt{2}} V^{00'} + V^{11'} = k(|\alpha^0|^2 + |\alpha^1|^2)$$

so $k \in \mathbb{R}$. If V^a is future pointing rescaling $\alpha^A \longmapsto \sqrt{k}\alpha^A$ yields the result. Conversely if $k = 1$ then clearly $V^0 > 0$. □

Remark 4.22. We see that a spinor o^A defines a future pointing null vector via $V^a = o^A \overline{o}^{A'}$ while a future pointing null vector defines a spinor up to phase.

Lemma 4.23. $\Psi(\eta_{ab}) = \epsilon_{AB}\epsilon_{A'B'}$. In particular $V^{AA'} V_{AA'} = V^a V_a$.

Proof. Using the combinatorial definition of the determinant we find

$$||V^a||^2 = 2\det(V^{AA'}) = \epsilon_{AB}\epsilon_{A'B'} V^{AA'} V^{BB'}$$

Therefore $\epsilon_{AB}\epsilon_{A'B'}$ yields the correct norms. It trivially satisfies the parallelogram law, so we are done. □

Lemma 4.24. Suppose we have a spinor $\alpha_{\ldots CD \ldots} = \alpha_{\ldots [CD] \ldots}$. Then

$$\alpha_{\ldots CD \ldots} = \frac{1}{2} \epsilon_{CD} \alpha_{\ldots A}{}^{A}{}_{\ldots}$$

Hence any spinor may be written as a product of a symmetric spinor and the distinguished antisymmetric spinors ϵ.

Proof. Note that $\epsilon_{A[B}\epsilon_{CD]} = 0$ for it is skew on 3 indices. Writing this out fully and raising some indices yields the identity

$$\epsilon_{AB}\epsilon^{CD} = \delta_A^C \delta_B^D - \delta_A^D \delta_B^C$$

The result follows immediately by applying this. □

Definition 4.25. A **bivector** is an antisymmetric valence 2 tensor $F^{ab} = F^{[ab]}$. We say that a bivector is **simple** if there exist vectors V^a and W^b with $F = V \wedge W$.

Remark 4.26. Beware the discrepancy between the definitions of simple for bivectors and general tensors.

Definition 4.27. The **Hodge dual** of a bivector F on Minkowski space is defined as

$$(*F)_{ab} = \frac{1}{2} \varepsilon_{ab}{}^{cd} F_{cd}$$

where ε_{abcd} is the Levi-Civita symbol.

Remark 4.28. A standard calculation shows that $**F = -F$. Hence $*$ has eigenspaces with eigenvalues $\pm i$. We say a bivector F is **self-dual** (SD) if $*F = iF$ and **anti-self-dual** (ASD) if $*F = -iF$. Any bivector can be decomposed into SD and ASD parts via

$$F = \frac{1}{2}(F - i*F) + \frac{1}{2}(F + i*F)$$

Lemma 4.29. $\Psi(\varepsilon_{abcd}) = i(\epsilon_{AC}\epsilon_{BD}\epsilon_{A'D'}\epsilon_{B'C'} - \epsilon_{AD}\epsilon_{BC}\epsilon_{A'C'}\epsilon_{B'D'})$.

Proof. Clearly the RHS is antisymmetric when any two unprimed indices and their corresponding primed indices are interchanged. Furthermore the RHS is Hermitian, so certainly the result is true up to scale. Now a simple calculation shows that $\varepsilon^{abcd}\varepsilon_{abcd} = -4! = -24$. The corresponding calculation on the RHS yields $-(2^4 \times 2 - 4 \times 2) = -24$ which establishes the claim. $\square$

Theorem 4.30. If F_{ab} is a real bivector then

$$F_{ab} = \varphi_{AB}\epsilon_{A'B'} + \overline{\varphi}_{A'B'}\epsilon_{AB}$$

holds for some symmetric spinor φ_{AB}, and is a decomposition of F into ASD and SD parts respectively.

Proof. We decompose $F_{ab} = F_{ABA'B'}$ as follows

$$\begin{aligned} F_{ABA'B'} &= F_{(AB)A'B'} + F_{[AB]A'B'} \\ &= F_{(AB)A'B'} + \frac{1}{2}\epsilon_{AB}F_C{}^C{}_{A'B'} \\ &= F_{(AB)(A'B')} + \frac{1}{2}\epsilon_{A'B'}F_{(AB)C'}{}^{C'} + \frac{1}{2}\epsilon_{AB}F_C{}^C{}_{(A'B')} + \frac{1}{4}\epsilon_{A'B'}F_{CC'}{}^{CC'} \end{aligned}$$

Since F is a bivector only the middle terms are nonzero. Moreover F is Hermitian so we may write

$$F_{ABA'B'} = \epsilon_{A'B'}\varphi_{AB} + \epsilon_{AB}\overline{\varphi}_{A'B'}$$

Using the previous lemma we find

$$\varepsilon_{ab}{}^{cd} = i(\delta_A^C\delta_B^D\delta_{A'}^{D'}\delta_{B'}^{C'} - \delta_A^D\delta_B^C\delta_{A'}^{C'}\delta_{B'}^{D'})$$

whence $\varphi_{AB}\epsilon_{A'B'}$ is ASD and $\overline{\varphi}_{A'B'}\epsilon_{AB}$ is SD. $\square$

Lemma 4.31. A symmetric valence n spinor may be factorised as

$$\varphi_{A\ldots B} = \alpha_{(A}\ldots\beta_{B)}$$

The null vectors defined by the spinors $\alpha_A, \dots \beta_B$ are called the **principal null directions** (PNDs) of $\varphi_{A\dots B}$.

Proof. Fix some basis for S and define $\xi^A = (1, x)$. Consider the scalar function of x given by $\varphi_{A\dots B}\xi^A \dots \xi^B$. It is a polynomial of degree n, and since $x \in \mathbb{C}$ we may factorise. Choose spinors $\alpha_A, \dots \beta_A$ such that the factors are $\alpha_A\xi^A, \dots \beta_A\xi^A$. Note that these are only defined up to scale. Multiplying out the factors and equating coefficients yields the result. $\square$

Definition 4.32. A **spinor field** on a four-dimensional Lorentzian manifold $\mathcal{M}$ is a smooth choice of spinor for each $x \in M$. Equivalently it is a smooth section of a (tensor product of) spin bundle(s) over $\mathcal{M}$ as defined in Example 3.63. We write $\mathcal{S}$ for the space of valence 1 unprimed spinor fields over $\mathcal{M}$.

Definition 4.33. A **spinor covariant derivative** on $\mathcal{M}$ is a map $\nabla_{AA'} : \mathcal{S} \longrightarrow \mathcal{S}^* \otimes \mathcal{S}'^* \otimes \mathcal{S}$ satisfying

1. $\nabla_{AA'}(\alpha^B + \beta^B) = \nabla_{AA'}\alpha^B + \nabla_{AA'}\beta^B$
2. $X^{AA'}\nabla_{AA'}f = X(f)$
3. $\nabla_{AA'}(f\alpha^B) = f\nabla_{AA'}\alpha^B + \alpha^B\nabla_{AA'}f$

for all spinor fields α^B, β^B, scalar fields f and complex vector fields X^a on $\mathcal{M}$. We define $\nabla_{AA'} : \mathcal{S}^* \longrightarrow \mathcal{S}^* \otimes \mathcal{S}'^* \otimes \mathcal{S}^*$ by requiring the Leibniz rule to hold, namely

$$\nabla_{AA'}(\alpha^B\beta_B) = \nabla_{AA'}(\alpha^B)\beta_B + \alpha^B\nabla_{AA'}(\beta_B)$$

We define $\nabla_{AA'} : \mathcal{S}' \longrightarrow \mathcal{S}^* \otimes \mathcal{S}'^* \otimes \mathcal{S}'$ by complex conjugation, namely

$$\nabla_{AA'}\alpha^{B'} = \overline{\nabla_{AA'}\overline{\alpha}^B}$$

We define the covariant derivative of a general spinor field by requiring that the Leibniz rule hold on all contracted products $\chi^{A\dots B'}{}_{C\dots D'}\alpha_A \dots \beta_{B'}\gamma^C \dots \delta^{D'}$.

Remark 4.34. Observe immediately that $\nabla_{AA'}$ is real, in the sense that $\overline{\nabla_{AA'}\alpha^{B'}} = \nabla_{AA'}\overline{\alpha^{B'}}$. Moreover $\nabla_{AA'}$ clearly commutes with contraction.

Theorem 4.35. There is a unique torsion free spinor covariant derivative on $\mathcal{M}$ satisfying $\nabla_{AA'}\epsilon_{BC} = 0$.

Proof. For uniqueness we refer the interested reader to Penrose and Rindler [33, p. 214]. We explicitly construct such a covariant derivative. Let ∇_a be the metric covariant derivative on $\mathcal{M}$, extended to complex vector fields via $\nabla_a(X^b + iY^b) = \nabla_aX^b + i\nabla_aY^b$. This defines $\nabla_{AA'}$ on spinor fields with equal numbers of primed and unprimed indices both in lower and upper position.

We extend this to general spinor fields using the Leibniz rule again. Fix an arbitrary spinor field α^B. Define a map $f : \mathcal{S}' \times \mathcal{S}' \longrightarrow \mathcal{S}^* \otimes \mathcal{S}'^* \otimes \mathcal{S}$ by

$$f(\xi^{B'}, \eta^{C'}) = \xi_{B'} \nabla_a (\alpha^B \eta^{B'}) + \eta_{B'} \nabla_a (\alpha^B \xi^{B'}) - \alpha^B \nabla_a (\xi_{B'} \eta^{B'})$$

which is well defined since ∇_a acts only on complex vector and scalar fields. An easy calculation shows that this is a $\Gamma(\mathcal{M})$-bilinear map, so may be written

$$f(\xi^{B'}, \eta^{C'}) = \theta_a{}^B{}_{B'C'}(\alpha^B)\xi^{B'}\eta^{C'}$$

Observe that $\theta_a{}^B{}_{B'C'}$ must be antisymmetric in $B'C'$ so we have

$$\theta_a{}^B{}_{B'C'} = \varphi_a{}^B(\alpha^B)\epsilon_{B'C'}$$

Finally we define $\nabla_{AA'}\alpha^B = \frac{1}{2}\varphi_{AA'}{}^B$ so in particular

$$f(\xi^{B'}, \eta^{C'}) = 2(\nabla_{AA'}\alpha^B)\xi_{B'}\eta^{B'}$$

so the Leibniz rule holds as required. Now it is easily verified that $\nabla_{AA'}$ satisfies the definition of a covariant derivative. Tedious algebra establishes that $\nabla_{AA'}\epsilon_{BC} = 0$ and $\nabla_{AA'}$ is torsion free, cf. Penrose and Rindler [33, p. 218]. □

Remark 4.36. Henceforth we shall always assume that $\nabla_{AA'}$ is the metric spinor covariant derivative constructed above.

Example 4.37. We briefly describe a beautiful geometric interpretation of the correspondence between spinors and past-pointing null vectors. This will help to inform our intuition when we describe the twistor correspondence in §5.1.

Observe that $\mathbb{P}^1$ is the space of spinors at x regardless of scale and phase. This corresponds to the projective past null cone of an observer at x. Topologically $\mathbb{P}^1$ is a sphere, which we may interpret as the **celestial sphere** familiar from the night sky.

We may go further, and identify the effect of a Lorentz transformation in M on the celestial sphere $\mathbb{P}^1$. In particular we exhibit an isomorphism between the proper orthochronous Lorentz group $SO(1,3)^+$ and the conformal group $C(2)$ of S^2 (cf. §4.4). Denote the past null cone by N^- and write the Minkowski metric in spherical coordinates as

$$ds^2 = dt^2 - dr^2 - r^2(d\theta^2 - \sin^2\theta d\varphi^2)$$

so that the degenerate metric on N^- becomes

$$ds^2 = -r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

Suppose $r = R_1(\theta, \varphi)$ and $r = R_2(\theta, \varphi)$ are two cuts of the past null cone defining representatives for $\mathbb{P}N^-$. Then clearly the induced metrics are conformally related. Conversely we may obtain all metrics conformally related to the unit round sphere metric on $\mathbb{P}^1$ by taking some cut of the past null cone.

Denote by $\mathcal{C}$ the set of smooth cuts of N^-. We identify the conformal group of $\mathbb{P}^1$ as the group of transformations $\mathcal{C} \longrightarrow \mathcal{C}$ under composition. Now recall that proper orthchronous Lorentz transformations precisely map the null cone smoothly onto itself without reversing the arrow of time. Therefore we immediately have $SO(1,3)^+ = C(2)$.

As a physical application, consider the appearance of a spherical comet A moving past Earth at a relativistic speed. Naively, in the rest frame of Earth the comet should appear Lorentz contracted. However, such an effect is not observed. This is easily explained in the conformal picture.

A spherical comet B at rest relative to Earth intersects a circular cone of past null geodesics for a terrestrial observer O. Therefore B describes a circular disc on the celestial sphere of O. Under a Lorentz transformation B is mapped to A. The celestial sphere undergoes the corresponding conformal transformation, which sends circles to circles. Thus A also describes a circular disc on the celestial sphere of O.

4.2 Zero Rest Mass Fields

Definition 4.38. The **helicity operator** h on a particle state is defined as the projection of the spin operator $\mathbf{s}$ along the direction of the momentum operator $\mathbf{p}$. Mathematically we write $h = (\mathbf{p}.\mathbf{s})/|\mathbf{p}|$.

Remark 4.39. Helicity is a good quantum number for massless fields, since we cannot boost to a frame which changes the sign of the momentum.

Definition 4.40. We define the **Weyl equations** for spinor fields ψ_R, ψ_L on M by

$$\overline{\sigma}^\mu \partial_\mu \psi_R = 0 \text{ and } \sigma^\mu \partial_\mu \psi_L = 0$$

where $\sigma^\mu = (1, \sigma^i)$ and $\overline{\sigma}^\mu = (1, -\sigma^i)$. These describe massless non-interacting fermion fields.

Lemma 4.41. ψ_R has helicity $+1/2$ and ψ_L has helicity $-1/2$.

Proof. Fourier transforming the first equation we obtain

$$\sigma^i p_i \psi_R(p) = E\psi_R(p)$$

Since $m = 0$ we have $E = |\mathbf{p}|$ and thus

$$(\boldsymbol{\sigma}.\mathbf{p})/|\mathbf{p}|\psi_R(p) = \psi_R(p)$$

Recall that for spin 1/2 particles we define $\mathbf{S} = \boldsymbol{\sigma}/2$ whence

$$h\psi_R(p) = \frac{1}{2}\psi_R(p)$$

as required. The negative helicity case follows similarly. $\square$

Lemma 4.42. The Weyl equations may equivalently be written

$$\nabla_{AA'}\alpha^A = 0 \text{ and } \nabla^{AA'}\beta_{A'} = 0$$

where α^A has helicity $-1/2$ and $\beta_{A'}$ has helicity $+1/2$.

Proof. By convention we choose $\alpha = \psi_L \in \mathcal{S}$ and $\beta = \psi_R \in \mathcal{S}'^*$. Now recall that $\nabla^{AA'} = \sum \sigma^a \nabla^a$ and $\nabla_{AA'} = \sum \sigma^a \nabla_a$. The result follows easily. $\square$

Definition 4.43. We define **Maxwell's equations** for a bivector field F on M by

$$dF^+ = 0 \text{ and } dF^- = 0$$

where F^+ is the SD and F^- the ASD part of F. These describe a massless non-interacting source-free electromagnetic field.

Remark 4.44. We note without proof that F^+ describes a field of helicity $+1$, while F^- describes a field of helicity -1. For a full development of the interaction between duality and helicity, see Bialynicki-Birula et al. [6].

Lemma 4.45. Maxwell's equations may equivalently be written

$$\nabla^{AA'}\varphi_{AB} = 0 \text{ and } \nabla^{AA'}\varphi_{A'B'} = 0$$

where φ_{AB} has helicity -1 and $\varphi_{A'B'}$ has helicity $+1$.

Proof. An easy calculation shows that Maxwell's equations are equivalent to

$$\nabla^a F^+_{ab} = 0 \text{ and } \nabla^a F^-_{ab} = 0$$

Now write $F_{ab} = \varphi_{AB}\epsilon_{A'B'} + \varphi_{A'B'}\epsilon_{AB}$ and $\nabla^a = \nabla^{AA'}$ and we're done. $\square$

Definition 4.46. We define the **zero rest mass** (ZRM) equations for symmetric valence n spinor fields $\varphi_{A\ldots B}$ and $\varphi_{A'\ldots B'}$ on M by

$$\begin{aligned} \nabla^{AA'}\varphi_{A\ldots B} &= 0 \quad \text{for helicity } -n/2 \\ \nabla^{AA'}\varphi_{A'\ldots B'} &= 0 \quad \text{for helicity } \quad n/2 \\ \nabla^{AA'}\nabla_{AA'}\varphi &= 0 \quad \text{for helicity } \quad 0 \end{aligned}$$

Remark 4.47. Following Huggett and Tod [21, p. 31] we assume without proof that these equations describe massless fields of the stated helicities. Indeed such a viewpoint is well motivated by our results for the Weyl and Maxwell equations. The diligent reader may wish to extend the proof of Lemma 4.41 to this general case, referring to the classic paper of Bargmann and Wigner [4].

Lemma 4.48. $\nabla_{BA'}\nabla^{AA'} = \frac{1}{2}\delta_B^A\Box$ in Minkowski space M.

Proof. For $A = B$ the result is obviously true. Suppose $A \neq B$, taking wlog $A = 1$ and $B = 0$. Then

$$\nabla_{BA'}\nabla^{AA'} = \nabla_{0A'}\nabla^{1A'} = \nabla_{00'}\nabla^{10'} + \nabla_{01'}\nabla^{11'} = \nabla^{11'}\nabla^{10'} - \nabla^{10'}\nabla^{11'}$$

which vanishes since partial derivatives commute in flat space. □

Theorem 4.49. Suppose $\varphi_{A\ldots K}$ satisfies the ZRM equations. Then $\Box\varphi_{AB\ldots K} = 0$. A similar result holds for positive helicity fields.

Proof. Using the previous lemma we find

$$0 = 2\nabla^{AA'}\varphi_{AB\ldots K} = 2\nabla_{MA'}\nabla^{AA'}\varphi_{AB\ldots K} = \delta_M^A\Box\varphi_{AB\ldots K} = \Box\varphi_{MB\ldots K}$$

which establishes the claim. □

Lemma 4.50. The zero rest mass field equations $\nabla^{AA'}\varphi_{AB\ldots C} = 0$ may equivalently be written $\nabla_{MM'}\varphi_{AB\ldots C} = \nabla_{M'(M}\varphi_{AB\ldots C)}$.

Proof. By the antisymmetry of ϵ we have

$$\begin{aligned} 0 = \nabla^{AA'}\varphi_{AB\ldots C} &= \epsilon^{MA}\epsilon^{M'A'}\nabla_{MM'}\varphi_{AB\ldots C} \\ \Leftrightarrow \ \nabla_{M'M}\varphi_{AB\ldots C} &= \nabla_{M'(M}\varphi_{A)B\ldots C} \end{aligned}$$

By definition φ is symmetric in its indices. Note that if a tensor is symmetric in two overlapping groups of indices then it is symmetric in the union of the groups, since every permutation is a product of transpositions. The result follows immediately. □

4.3 Hertz Potentials

Lemma 4.51. Suppose $\psi_{F\ldots K}^{M'\ldots Q'}$ is a ZRM field in its $(n-k)$ unprimed indices and symmetric in its k primed indices. Then

$$\varphi_{AB\ldots K} = \nabla_{AM'} \ldots \nabla_{EQ'} \psi_{F\ldots K}^{M'\ldots Q'}$$

defines a ZRM field with n indices.

Proof. Since we working in Minkowski space the derivatives commute, so $\varphi_{AB\ldots K}$ is certainly symmetric in $A \ldots E$. But by the previous lemma $\varphi_{AB\ldots K}$ is symmetric in $E \ldots K$, so $\varphi_{AB\ldots K}$ is totally symmetric. Moreover

$$\begin{aligned} 2\nabla^{AA'}\varphi_{AB\ldots K} &= 2\nabla^{AA'}\nabla_{AM'} \ldots \nabla_{EQ'} \psi_{F\ldots K}^{M'\ldots Q'} \\ &= \nabla_{BN'} \ldots \nabla_{EQ'} \delta_{M'}^{A'} \Box \psi_{F\ldots K}^{M'\ldots Q'} = 0 \end{aligned}$$

using Lemma 4.48 and Theorem 4.49. □

Theorem 4.52. Let $\varphi_{AB\ldots K}$ be a ZRM field with n indices. Fix $k \leq n$. Then locally there exists a spinor field $\psi_{F\ldots K}^{M'\ldots Q'}$ satisfying

1. $\psi_{F\ldots K}^{M'\ldots Q'}$ is a ZRM field in its $(n-k)$ unprimed indices
2. $\psi_{F\ldots K}^{M'\ldots Q'}$ is symmetric in its k primed indices
3. $\varphi_{AB\ldots K} = \nabla_{AM'} \ldots \nabla_{EQ'} \psi_{F\ldots K}^{M'\ldots Q'}$

We call $\psi_{F\ldots K}^{M'\ldots Q'}$ a **Hertz potential** for $\varphi_{AB\ldots K}$.

Proof. We do the case $k = 1$, then the general result follows inductively. Fix a point $x \in M$ and a constant spinor field $\mu^{M'}$ in a neighbourhood of x. Wlog work in a trivialisation for $\mathcal{S}'$ such that $\mu^{M'} = (1, 0)$. For convenience define

$$u^A = x^{A0'} \text{ and } v^A = x^{A1'}$$

so that $\mu^{M'}\nabla_{AM'} = \partial/\partial u^A$. With this notation the ZRM equations in the form of Lemma 4.50 become

$$\begin{aligned} \frac{\partial}{\partial u^Y}\varphi_{AB\ldots K} - \frac{\partial}{\partial u^A}\varphi_{YB\ldots K} &= 0 \qquad (\star) \\ \frac{\partial}{\partial v^Y}\varphi_{AB\ldots K} - \frac{\partial}{\partial v^A}\varphi_{YB\ldots K} &= 0 \end{aligned}$$

Now $(\star)$ states that 2-dimensional curl of $\varphi_{AB\ldots K}$ in the variables u^A vanishes. Therefore there exists a local potential $\chi_{B\ldots K}$ defined in some neighbourhood of x such that

$$\varphi_{AB\ldots K} = \frac{\partial}{\partial u^A}\chi_{B\ldots K} = \mu^{M'}\nabla_{AM'}\chi_{B\ldots K} \qquad (\dagger)$$

We set $\psi^{M'}_{B\ldots K} = \mu^{M'}\chi_{B\ldots K}$. Since $\varphi_{AB\ldots K}$ is symmetric, certainly $\chi_{B\ldots K}$ is symmetric. It remains to show that $\chi_{B\ldots K}$ can be chosen to satisfy the ZRM equations. The symmetry of $\varphi_{AB\ldots K}$ immediately implies

$$\frac{\partial}{\partial u^A}\chi_{BC\ldots K} - \frac{\partial}{\partial u^B}\chi_{AC\ldots K} = 0 \qquad (\diamond)$$

We also easily obtain

$$\frac{\partial}{\partial u^B}\left(\frac{\partial}{\partial v^Y}\psi_{AC\ldots K} - \frac{\partial}{\partial v^A}\psi_{YC\ldots K}\right) = 0$$

so the bracketed term $\theta_{YAC\ldots K}$ depends only on the variables v^A. Thus redefining $\psi_{0C\ldots K} \longmapsto \psi_{0C\ldots K} - \int \theta_{10C\ldots K}dv^1$ does not affect ($\diamond$) or ($\dagger$) and we may check

$$\frac{\partial}{\partial v^1}\psi_{0C\ldots K} - \frac{\partial}{\partial v^0}\psi_{1C\ldots K} = \theta_{10C\ldots K} - \theta_{10C\ldots K} = 0$$

as required. □

Remark 4.53. The deduction of ($\dagger$) from ($\star$) is slightly nontrivial. For real variables it is well-known that a vector field with vanishing curl has a scalar potential over any *simply connected* region of spacetime. However, $u^1 = x^{10'}$ is a complex variable. Fortunately there is a suitable modification of the theorem which works in our case. It only guarantees the existence of a potential on regions of spacetime with *vanishing second homotopy group*. See Penrose [29, p. 166] for details.

4.4 Conformal Invariance

We now exhibit an important property of the zero rest mass and Yang-Mills equations, namely their invariance under angle-preserving transformations of Minkowski space. This will provide ample motivation for accomodating the concept of conformal invariance at the heart of twistor theory. Indeed, Penrose and MacCallum [32] view a twistor as an element of a representation space for the universal cover of the conformal group. We shall not pursue this intepretation fully; nevertheless the role of conformal transformations is manifest in §5.

Definition 4.54. A **conformal map** of pseudo-Riemannian manifolds is a smooth map $f : (\mathcal{M}, g) \longrightarrow (\mathcal{N}, h)$ such that $f^*h = \Omega^2 g$ for some smooth function $\Omega : \mathcal{M} \longrightarrow \mathbb{R}$. We call Ω^2 the **conformal factor**.

Lemma 4.55. A smooth map $f : \mathcal{M} \longrightarrow \mathcal{N}$ of Riemannian manifolds is conformal iff it preserves angles between tangent vectors.

Proof. Let $x \in \mathcal{M}$ and choose $X, Y \in T_x\mathcal{M}$. Then the angle between them is given by

$$\cos\theta = \frac{g_x(X,Y)}{\sqrt{g_x(X,X)g_x(Y,Y)}}$$

The angle between their pushforwards is given by

$$\begin{aligned}
\cos\tilde{\theta} &= \frac{h_{f(x)}(f_*X, f_*Y)}{\sqrt{h_{f(x)}(f_*X, f_*X)h_{f(x)}(f_*Y, f_*Y)}} \\
&= \frac{(f^*h)_x(X,Y)}{\sqrt{(f^*h)_x(X,X)(f^*h)_x(Y,Y)}} \\
&= \frac{\Omega^2 g_x(X,Y)}{\Omega^2\sqrt{g_x(X,X)g_x(Y,Y)}} \\
&= \cos\theta
\end{aligned}$$

Restrict $\theta, \tilde{\theta} \in [-\pi, \pi)$ then both directions follow immediately. □

Remark 4.56. Therefore conformal maps change infinitesimal scale while preserving infinitesimal shape.

Definition 4.57. A **(global) conformal transformation** is a conformal diffeomorphism $f : (\mathcal{M}, g) \longrightarrow (\mathcal{M}, g)$.

Remark 4.58. Conformal transformations are essentially coordinate changes which preserve the metric components up to scale. In particular every isometry is a conformal transformation.

Definition 4.59. A **local conformal transformation** of $(\mathcal{M}, g)$ is a conformal diffeomorphism $f : (U, g) \longrightarrow (V, g)$ where U and V are open subsets of $\mathcal{M}$.

Example 4.60 (Dilations and Inversions in Minkowski Space). Let M be Minkowski space. Consider the diffeomorphism $f : M \longrightarrow M$ given by

$$f : x \longmapsto kx$$

where k is a nonzero real constant. The metric η pulls back to $k^2\eta$, so f is a conformal transformation of M, called a **dilation**.

Let o denote the origin of M and $p \in M$ be arbitrary. Define

$$\begin{aligned}
M_o &= M \setminus \{\text{light cone at } o\} \\
M_p &= M \setminus \{\text{light cone at } p\}
\end{aligned}$$

Consider the diffeomorphism $f : M_p \longrightarrow M_o$ given by

$$f : x \longmapsto \frac{p - x}{(p - x)^2}$$

The metric η pulls back to $(p - x)^{-4}\eta$, so f is a local conformal transformation of M, called an **inversion**.

Theorem 4.61 (Liouville)**.** Any local conformal transformation of Minkowski space is the composition of inversions, dilations and Poincaré transformations.

Proof. Consider an infinitesimal local conformal transformation of Minkowski space, and derive algebraic constraints on the parameters. The result quickly follows. See Di Francesco et al. [8, p. 96] for details. □

Remark 4.62. Since inversions are singular on a light cone, it is not always possible to extend a local conformal transformation to a global one. This is mathematically and physically inconvenient. We remedy this problem by defining a **conformal completion** $\tilde{M}$, in which every local conformal transformation has a unique global extension.

It turns out that Minkowski space has a natural compactification which is also a conformal completion. We present this construction in §5.3. We shall avoid a detailed discussion of the ambiguities involved in defining a conformal completion more generally, referring the interested reader to Geroch [15].

Definition 4.63. The **conformal group** of a manifold is the group of global conformal transformations of its conformal completion.

Definition 4.64. A **Weyl transformation** or **conformal rescaling** is a conformal diffeomorphism $\mathrm{id} : (\mathcal{M}, g) \longrightarrow (\mathcal{M}, h)$.

Remark 4.65. Weyl transformations may be viewed as active rescalings of the metric, independent of any coordinate change. Note that any conformal diffeomorphism $f : (\mathcal{M}, g) \longrightarrow (\mathcal{M}, h)$ is the composition of a conformal transformation and a Weyl transformation.

Definition 4.66. Two metrics g and h on $\mathcal{M}$ are **conformally equivalent** if there is a Weyl transformation $(\mathcal{M}, g) \longrightarrow (\mathcal{M}, h)$, that is, if $g = \Omega^2 h$ for some smooth $\Omega : \mathcal{M} \longrightarrow \mathbb{R}$. This defines an equivalence relation on the set of all metrics on $\mathcal{M}$. An equivalence class is called a **conformal class** or **conformal metric**.

Definition 4.67. A **conformal structure** on a pseudo-Riemannian manifold is a choice of conformal metric.

Remark 4.68. The conformal transformations are precisely those diffeomorphisms $f : (\mathcal{M}, g) \longrightarrow (\mathcal{M}, g)$ which preserve the conformal structure.

Definition 4.69. Let $(\mathcal{M}, g)$ be pseudo-Riemannian and $x \in \mathcal{M}$. The **light cone** at x is the set of $X \in T_x\mathcal{M}$ such that $g_x(X, X) = 0$. The **light cone structure** of $\mathcal{M}$ is the set of light cones at all $x \in \mathcal{M}$.

Theorem 4.70. Let $\mathcal{M}$ be a manifold of dimension n, and assume all metrics are Lorentzian. Specifying a conformal structure on $\mathcal{M}$ is equivalent to specifying a light cone structure on $\mathcal{M}$.

Proof. ($\Rightarrow$) Suppose that g and h are Lorentzian metrics in the same conformal class. We show that they give rise to the same light cone structure. Write $h = \Omega^2 g$, for some $\Omega^2 \neq 0$. Fix $x \in \mathcal{M}$ and $X \in T_x\mathcal{M}$. Then trivially $h_x(X, X) = 0$ iff $g_x(X, X) = 0$ as required.

($\Leftarrow$) Suppose that g and h are Lorentzian metrics giving rise to the same light structure. We show that they lie in the same conformal class. Fix $x \in X$ and suppose $g_x(X, X) = 0$ iff $h_x(X, X) = 0$ for all $X \in T_x M$. Let $\{e_1, \ldots, e_n\}$ be a g-orthonormal basis for $T_x\mathcal{M}$, with e_1 g-timelike and e_i g-spacelike for $i \neq 1$. By assumption we have $h_{jj} = h(e_j, e_j) \neq 0$ for all j.

Now fix any $j, k \geq 2$ with $j \neq k$. For $\theta \in \mathbb{R}$ consider the g-null vectors

$$V(\theta) = e_1 + e_j \cos\theta + e_k \sin\theta$$

By assumption we have

$$\begin{aligned} 0 &= h(V(\theta), V(\theta)) \\ &= h_{11} + h_{jj}\cos^2\theta + h_{kk}\sin^2\theta + 2h_{1j}\cos\theta + 2h_{1k}\sin\theta + h_{jk}\sin(2\theta) \qquad (\star) \end{aligned}$$

Now taking $\theta = 0, \pi$ we get

$$0 = h_{11} + h_{jj} + 2h_{1j} = h_{11} + h_{jj} - 2h_{1j}$$

whence we conclude that $h_{1j} = 0$ and $h_{jj} = -h_{11}$ for all $j \geq 2$. Then $(\star)$ becomes

$$h_{jk}\sin(2\theta) = 0$$

whence $h_{jk} = 0$ for all $j \neq k$ with $j, k \geq 2$. Now choosing $\Omega^2(x) = h_{11} > 0$ we find that $h_{\mu\nu} = \Omega^2 g_{\mu\nu}$ at x. But h and g are tensors so this result is true in any coordinate system. Since x was arbitrary, the proof is complete. □

Remark 4.71. Clearly the forward implication holds for arbitrary metrics. The backward direction can be proved for metrics of indefinite signature; see Ehrlich [11] for details.

Remark 4.72. This theorem rigorously explains the language of Ward and Wells [39, p. 51].

Definition 4.73. An equation on a manifold $(\mathcal{M}, g)$ is **conformally invariant** on $U \subset \mathcal{M}$ if it remains unchanged under the action of every local conformal transformation with domain U.

Lemma 4.74. *A Poincaré invariant field equation on Minkowski space (M, η) is locally conformally invariant if every field φ_i has a conformal weight r_i such that the equation is unchanged under the conformal rescaling $\eta \longmapsto \Omega^2\eta$ with $\varphi_i \longmapsto \Omega^{r_i}\varphi_i$ $\forall i$.*

Proof. Following a hint of Penrose and MacCallum [32, §1.1] we demonstrate that the Poincaré transformations of Minkowski space become conformal transformations according to any other conformally rescaled metric, and that the conformal transformations obtainable in this way generate the full conformal group $C(1,3)$. Then the result follows immediately from the assumptions of the lemma.

Consider a conformally rescaled metric $g_x = \Omega(x)\eta_x$ where $\Omega(x)$ is a smooth positive function on M. An infinitesimal Poincaré transformation takes the form

$$x^\mu \longmapsto x'^\mu = x^\mu + \omega^\mu_\nu x^\nu + \epsilon^\mu$$

where ω^μ_ν is an antisymmetric matrix. Under such a transformation Ω transforms as

$$\Omega'(x) = \Omega(x^\mu - \omega^\mu_\nu x^\nu - \epsilon^\mu) = \Omega(x) - \omega^\mu_\nu x^\nu \partial_\mu \Omega(x) - \epsilon^\mu \partial_\mu \Omega(x)$$

and so the metric changes by

$$\delta g_x = -\eta_x(\omega^\mu_\nu x^\nu + \epsilon^\mu)\partial_\mu \Omega(x)$$

which is certainly a conformal transformation with respect to g.

Now since g is conformally equivalent to η, there exists $\varphi \in C(1,3)$ such that $\varphi_* \eta = g$. Hence the subgroup of $C(1,3)$ which fixes g is isomorphic to the Poincaré group. Therefore $C(1,3)$ is generated by the Poincaré group and those transformations which effect an arbitrary conformal rescaling with respect to some conformally equivalent metric.

It only remains to show that we may choose Ω, ω and ϵ to make δg arbitrary. Equivalently, given a smooth $f : M \longrightarrow \mathbb{R}$ we must find a solution to

$$f(x)^2 = -(\omega^\mu_\nu x^\nu + \epsilon^\mu)\partial_\mu \Omega(x)$$

by varying Ω, ω and ϵ, which is always possible using the method of characteristics. $\square$

Remark 4.75. It is immediately apparent that massive theories are not conformally invariant. For indeed a massive field must obey the Klein-Gordon equation $(\square + m^2)\varphi = 0$. Now under a dilation $\eta \longmapsto k^2\eta$ the D'Alembertian transforms as $\square \longmapsto k^{-2}\square$, so clearly there is no appropriate conformal weight unless $m = 0$.

Theorem 4.76. The zero rest mass field equations are conformally invariant on any open subset of Minkowski space M.

Proof. Consider a conformal rescaling $\eta_{ab} \longmapsto \hat{\eta}_{ab} = \Omega^2\eta_{ab}$. We are interested in finding how the covariant derivative $\nabla_{AA'}$ transforms. Inspired by the identity $\eta_{ab} = \epsilon_{AB}\epsilon_{A'B'}$ we choose

$$\hat{\epsilon}_{AB} = \Omega\epsilon_{AB},\ \hat{\epsilon}_{A'B'} = \Omega\epsilon_{A'B'}$$

Recall from §4.1 that we must have $\hat{\nabla}_{AA'}\epsilon_{BC} = 0$. Moreover the covariant derivative of a scalar field f is independent of connection, so in particular

$$\nabla_{AA'}f = \hat{\nabla}_{AA'}f$$

Therefore the action of $\hat{\nabla}_{AA'} - \nabla_{AA'}$ on spinors ξ^C is $\Gamma(M)$-linear so we may write

$$\hat{\nabla}_{AA'}\xi^C = \nabla_{AA'}\xi^C + \Gamma^C_{AA'B}\xi^B$$

Recall that we require ∇ and $\hat{\nabla}$ to be torsion free. This condition implies (cf. Penrose and Rindler [33, p. 217])

$$\Gamma^C_{AA'B} = i\Pi_{AA'}\epsilon^C_B + \Upsilon_{A'B}\epsilon^C_A$$

where Υ_a and Π_a are real vector fields on M. Now we may calculate

$$\begin{aligned} 0 &= \hat{\nabla}_{AA'}\hat{\epsilon}_{BC} = \hat{\nabla}_{AA'}(\Omega\epsilon_{BC}) \\ &= \epsilon_{BC}(\nabla_{AA'}\Omega - 2i\Omega\Pi_{AA'} - \Omega\Upsilon_{AA'}) \end{aligned}$$

by applying the defining properties of a connection. Rearranging we get

$$\Omega^{-1}\nabla_a\Omega = \Upsilon_a + 2i\Pi_a$$

Since Ω is real we must take $\Pi_a = 0$ and $\Upsilon_a = \Omega^{-1}\nabla_a\Omega$ whence

$$\Gamma^C_{AA'B} = \epsilon^C_A\Omega^{-1}\nabla_{A'B}\Omega$$

Then by the Leibniz rule we have

$$\begin{aligned}\hat{\nabla}_{AA'}\varphi_{C\dots F} &= \nabla_{AA'}\varphi_{C\dots F} - \Gamma^{B}_{AA'C}\varphi_{BD\dots F} - \cdots - \Gamma^{B}_{AA'F}\varphi_{C\dots EB} \\ &= \nabla_{AA'}\varphi_{C\dots F} - \Omega^{-1}(\nabla_{A'C}\Omega)\varphi_{AD\dots F} - \cdots - \Omega^{-1}(\nabla_{A'F}\Omega)\varphi_{C\dots EA}\end{aligned}$$

Recall from Lemma 4.50 that we may write the ZRM equations as

$$\nabla_{AA'}\varphi_{C\dots F} = \nabla_{A'(A}\varphi_{C\dots F)}$$

Suppose $\varphi_{C\dots F}$ has conformal weight -1. Then we find

$$\begin{aligned}\hat{\nabla}_{AA'}\hat{\varphi}_{C\dots F} &= \hat{\nabla}_{AA'}(\Omega^{-1}\varphi_{C\dots F}) \\ &= \nabla_{AA'}(\Omega^{-1}\varphi_{C\dots F}) - \Omega^{-1}(\nabla_{A'C}\Omega)\Omega^{-1}\varphi_{AD\dots F} - \cdots \\ &\quad -\Omega^{-1}(\nabla_{A'F}\Omega)\Omega^{-1}\varphi_{C\dots EA} \\ &= \Omega^{-1}\nabla_{AA'}\varphi_{C\dots F} - \Omega^{-2}(\nabla_{AA'}\Omega^{-1})\varphi_{C\dots F} \\ &\quad -\Omega^{-2}(\nabla_{A'C}\Omega)\varphi_{AD\dots F} - \cdots - \Omega^{-2}(\nabla_{A'F}\Omega)\varphi_{C\dots EA}\end{aligned}$$

By construction the RHS less the first term is automatically symmetric in $\{A, C, \dots F\}$. Therefore the ZRM equations hold for (∇, φ) iff they hold for $(\hat{\nabla}, \hat{\varphi})$. The result for positive helicity fields $\varphi_{A'D'\dots F'}$ follows similarly. □

Remark 4.77. It is also possible to show that the wave equation is conformally invariant, for which we refer the interested reader to Ward and Wells [39, p. 291].

Theorem 4.78. The anti-self-dual Yang-Mills equations are conformally invariant on any open subset of Minkowski space M.

Proof. We show that the Hodge star operator is conformally invariant. Let $V_{abcd} = \sqrt{|\det \eta|}\varepsilon_{abcd}$ be the volume form of M, where ε the Levi-Civita symbol. Under a conformal rescaling $\eta \longmapsto \Omega^2\eta$ we have $V \longmapsto \Omega^4 V$. Now recall

$$(*F)_{\mu\nu} = \frac{1}{2}V_{\mu\nu\rho\sigma}\eta^{\rho\tau}\eta^{\sigma\upsilon}F_{\tau\upsilon}$$

so $*F$ remains unchanged under a conformal rescaling. □

Remark 4.79. In fact it's easy to show that the full Yang-Mills equations are conformally invariant, but we shall not consider these here. See Ward and Wells [39, p. 292] for details.

4.5 Twistors From Dynamics

Definition 4.80. Let $x^a(t)$ be a particle trajectory in M, parameterised by coordinate time t. We define the particle 4**-momentum** by

$$p^a = (E, \mathbf{p}), \quad \mathbf{p} \propto \dot{\mathbf{x}}(t), \quad \mathbf{p}^2 = E^2 - m^2$$

where m denotes the particle mass. We define the particle **orbital angular momentum** by

$$J^{ab} = p^a \wedge x^b$$

We define the particle **total angular momentum** by

$$M^{ab} = J^{ab} + S^{ab}$$

where S^{ab} denotes the particle spin angular momentum.

Lemma 4.81. Let u^a and v^a be two orthogonal null vectors. Then $u^a = kv^a$ for some $k \in \mathbb{R}$.

Proof. Work in coordinates where $u^a = (A, 0, 0, A)$ and $v^a = (B, \mathbf{p})$. Then $u^a v_a = 0$ yields $p^3 = B$, so $p^1 = p^2 = 0$. Choose $k = A/B$. □

Definition 4.82. The **Pauli-Lubanski vector** is defined by

$$S^a = (*M)^{ab} p_b$$

Lemma 4.83. For a massless particle $S^a = hp^a$ where h is the particle helicity. Therefore the Pauli-Lubanski vector classically encodes the helicity of a massless particle.

Proof. Observe that

$$S^0 = \mathbf{S}.\mathbf{p} = h|\mathbf{p}| = hE$$

We easily check that S^a and p^a are orthogonal null vectors, and the result follows by the previous lemma. □

Lemma 4.84. Consider a massless particle with momentum p^a and total angular momentum M^{ab}. Write $p_a = \overline{\pi}_A \pi_{A'}$ and $M_{ab} = \mu_{AB}\epsilon_{A'B'} + \overline{\mu}_{A'B'}\epsilon_{AB}$. Then there exists a spinor ω_A such that $\mu_{AB} = -i\omega_{(A}\overline{\pi}_{B)}$.

Proof. In spinor notation the Pauli-Lubanski vector becomes

$$S_{DD'} = -i\overline{\pi}_D \pi^{A'} \overline{\mu}_{A'D'} + i\overline{\pi}^A \mu_{AD} \pi_{D'} = h\overline{\pi}_D \pi_{D'}$$

using the previous lemma. Contracting both sides with $\overline{\pi}^D$ gives

$$\mu_{AB}\overline{\pi}^A\overline{\pi}^B = 0$$

Now μ_{AB} is a symmetric spinor, so by Lemma 4.31 it factorises as

$$\mu_{AB} = \alpha_{(A}\beta_{B)}$$

Substituting we find

$$\begin{aligned} & \alpha_A\beta_B\overline{\pi}^A\overline{\pi}^B = -\alpha_B\beta_A\overline{\pi}^A\overline{\pi}^B \\ \Rightarrow \quad & \alpha_A\beta_B\overline{\pi}^A\overline{\pi}^B = 0 \\ \overset{\text{wlog}}{\Rightarrow} \quad & \qquad \beta_B \propto \overline{\pi}_B \end{aligned}$$

and the result follows. □

Definition 4.85. Let x^a describe a massless particle, with (p^a, M^{ab}) encoded by the spinor pair $(\omega^A, \pi_{A'})$. We call $(\omega^A, \pi_{A'})$ a **twistor** and denote it Z^α for $\alpha = 0, 1, 2, 3$. We define the **dual twistor** $Z_\alpha = (\overline{\pi}_A, \overline{\omega}^{A'})$.

Remark 4.86. Under the formal twistor correspondence of §5 the reader may check that the change in Z^α effected by a change in origin of M is consistent with the changes in M^{ab} and p^a viz.

$$\begin{aligned} x^a &\longmapsto x^a + q^a \\ p^a &\longmapsto p^a \\ M^{ab} &\longmapsto M^{ab} + p^a q^b - p^b q^a \end{aligned}$$

Lemma 4.87. A twistor Z^α also encodes the helicity of a massless particle via

$$h = \frac{1}{2} Z_\alpha Z^\alpha$$

Proof. Substituting we verify that

$$\begin{aligned} S_{DD'} &= -i\overline{\pi}_D\pi^{A'}\left(i\overline{\omega}_{(A'}\pi_{D')}\right) + i\overline{\pi}^A\pi_{D'}\left(-i\omega_{(A}\overline{\pi}_{D)}\right) \\ &= \frac{1}{2}\overline{\pi}_D\pi_{D'}\left(\overline{\omega}_{A'}\pi^{A'} + \omega_A\overline{\pi}^A\right) \\ &= \frac{1}{2}\overline{\pi}_D\pi_{D'}Z^\alpha Z_\alpha \end{aligned}$$

and the result follows. □

5 Twistor Geometry

In §4.5 we tentatively defined a twistor as a complex quantity encoding the momentum, angular momentum and helicity of a massless particle. Although this perspective is physically valuable, we shall now pursue a more powerful abstract approach.

In this section we see that twistors form a complex twistor space related to Minkowski space according to simple rules. The translation of geometrical objects between these two perspectives is known as the twistor correspondence. Although this is easy to notate, it is conceptually challenging.

We start with an informal discussion of twistors and their relation to Minkowski space. We develop a baby version of the twistor correspondence, good enough for most applications, but slightly imprecise. To establish a geometrical intuition we describe a concrete interpretation in terms of Robinson congruences.

We then move towards a fuller account of the twistor correspondence. We conformally compactify Minkowski space, and discuss the role of flag manifolds. Finally we regain the baby version of our results by choosing an appropriate coordinate chart. This material is not immediately essential, so the reader may omit it, referring back when necessary later in the text.

5.1 The Baby Twistor Correspondence

Definition 5.1. Twistor space T is a 4-dimensional complex vector space with elements Z^α ($\alpha = 0, 1, 2, 3$) and a Hermitian inner product

$$\Sigma(Z, W) = Z^0\overline{W^2} + Z^1\overline{W^3} + Z^2\overline{W^0} + Z^3\overline{W^1}$$

with respect to some fixed basis. Each element $Z^\alpha \in T$ is called a **twistor**.

We coordinatise T by a pair of spinors according to the isomorphism

$$T = S \otimes S'$$

writing $Z^\alpha = (\omega^A, \pi_{A'})$.

We identify the dual twistor space T^* with the conjugate twistor space $\overline{T}$ via $Z_\alpha = \Sigma_{\alpha\beta}\overline{Z^\beta}$ so that $\Sigma(Z, W) = Z^\alpha W_\alpha$. Explicitly we have

$$Z_0 = \overline{Z^2},\ Z_1 = \overline{Z^3},\ Z_2 = \overline{Z^0},\ Z_3 = \overline{Z^1}$$

so that $Z_\alpha = (\overline{\pi}_A, \overline{\omega}^{A'})$.

Remark 5.2. Recall that a Hermitian form is determined by its signature up to change of basis. It is not hard to verify that Σ has neutral signature $(++--)$.

This explains the language of Ward and Wells [39, p. 52].

Definition 5.3. We divide T into regions T^+, T^- and N accordingly as $\Sigma(Z, Z) > 0$, < 0 and $= 0$. A twistor $Z^\alpha \in N$ is called **null**.

Remark 5.4. Observe that these are well-defined since the quadratic form induced by a Hermitian form is always real.

Definition 5.5. Let A and B be sets. A **correspondence** $\mathscr{C} : A \longrightarrow B$ is an assignment to each point $a \in A$ a subset $\mathscr{C}(a) \subset B$. We say that $a \in A$ and $b \in B$ are **incident** iff $b \in \mathscr{C}(a)$ or equivalently $a \in \mathscr{C}^{-1}(b)$. The correspondence $\mathscr{C}$ is hence also called an **incidence relation**.

Definition 5.6. Complexified Minkowski space $\mathbb{C}M$ is the complexification of M equipped with the complex-linear extension of η.

Definition 5.7. Projective twistor space $\mathbb{P}T$ is the projectification of T. We call elements of $\mathbb{P}T$ **projective twistors**, but occasionally abuse nomenclature by referring to them simply as twistors.

Definition 5.8. We define the **twistor correspondence** $\mathscr{C} : \mathbb{C}M \longrightarrow T$ by specifying that a twistor $Z^\alpha = (\omega^A, \pi_{A'})$ is incident with a spacetime point $x^a = x^{AA'}$ iff

$$\omega^A = ix^{AA'}\pi_{A'}$$

Note immediately that this descends to a correspondence $\mathscr{C} : \mathbb{C}M \longrightarrow \mathbb{P}T$, which we shall also refer to as the twistor correspondence.

Conjugating and using the identification of $\overline{T}$ with T^* we obtain a dual correspondence. Explicitly, $x^{AA'}$ is incident with $Z_\alpha = (\omega_A, \pi^{A'})$ iff

$$\pi^{A'} = -ix^{AA'}\omega_A$$

Definition 5.9. A surface S in a (complexified) spacetime $\mathcal{M}$ is called **totally null** if every tangent vector to S is null.

Definition 5.10. An **α-plane** in $\mathbb{C}M$ is a totally null 2-plane such that every tangent bivector is self-dual.

Theorem 5.11. Let $[Z^\alpha] = [\omega^A, \pi_{A'}] \in \mathbb{P}T$. $\mathscr{C}^{-1}([Z^\alpha])$ is an α-plane in $\mathbb{C}M$.

Proof. Suppose $x^{AA'} \in \mathscr{C}^{-1}([Z^\alpha])$. Then

$$\mathscr{C}^{-1}([Z^\alpha]) = \{x^{AA'} + y^{AA'} : iy^{AA'}\pi_{A'} = 0\}$$

For fixed A the equation $y^{AA'}\pi_{A'} = 0$ implies that $y^{AA'} \propto \pi^{A'}$. Therefore the most general solution is $iy^{AA'} = \lambda^A\pi^{A'}$ for arbitrary λ^A, whence

$$\mathscr{C}^{-1}([Z^\alpha]) = \{x^{AA'} + \lambda^A\pi^{A'} : \lambda^A \in S\}$$

Clearly this defines a 2-plane P in $\mathbb{C}M$. Moreover $\lambda^A \pi^{A'}$ is a rank-1 matrix, so every tangent to P is null. Finally consider a general tangent bivector

$$F^{AA'BB'} = \lambda^A \pi^{A'} \tilde{\lambda}^B \pi^{B'} - \lambda^A \pi^{B'} \tilde{\lambda}^B \pi^{A'} - \lambda^B \pi^{A'} \tilde{\lambda}^A \pi^{B'} + \lambda^B \pi^{B'} \tilde{\lambda}^A \pi^{A'}$$

The first two terms cancel by symmetry of $\pi^{A'} \pi^{B'}$ leaving

$$F^{AA'BB'} = \pi^{B'} \pi^{A'} \lambda^{[B} \tilde{\lambda}^{A]} = k \pi^{A'} \pi^{B'} \epsilon^{AB}$$

for some $k \in \mathbb{C}$, so $F^{AA'BB'}$ self-dual as required. □

Remark 5.12. A similar argument shows that a point in $\mathbb{P}T^*$ corresponds to a totally null ASD 2-plane in $\mathbb{C}M$, called a **β-plane**.

Lemma 5.13. Let $x^{AA'} \in \mathbb{C}M$. $\mathscr{C}(x^{AA'}) = \mathbb{P}^1 \subset \mathbb{P}T$, a projective line.

Proof. $\mathscr{C}(x^{AA'}) = \{[\omega^A, \pi_{A'}] \in \mathbb{P}T : \omega^A = i x^{AA'} \pi_{A'}\}$ is completely determined by $\{[\pi_{A'}] : \pi_{A'} \in S'^* = \mathbb{C}^2\}$. □

Remark 5.14. We can therefore summarise the twistor correspondence geometrically as follows

$$\begin{array}{rcl} \mathbb{C}M & \longleftrightarrow & \mathbb{P}T \\ \text{point } x^a & \longleftrightarrow & \text{projective line } \mathbb{P}^1 \\ \alpha\text{-plane} & \longleftrightarrow & \text{point } [Z^\alpha] \end{array}$$

By applying the correspondence both ways we also see

$$\text{2 points lie on same } \alpha\text{-plane} \quad \longleftrightarrow \quad \begin{array}{c} \text{2 projective lines intersect} \\ \text{in a projective twistor} \end{array}$$

Unfortunately these relations do not clearly elucidate the relationship between $\mathbb{P}T$ and real Minkowski space M. We now consider this problem seriously.

5.2 Robinson Congruences

Lemma 5.15. If $Z^\alpha = (\omega^A, \pi_{A'})$ is a null twistor then the corresponding α-plane contains some real point $x_0^{AA'}$.

Proof. The condition $Z^\alpha Z_\alpha = 0$ may be written

$$\omega^A \overline{\pi}_A = -\overline{\omega}^{A'} \pi_{A'} = -\overline{\omega^A \overline{\pi}_A} \qquad (\star)$$

We therefore have $\omega^A \overline{\pi}_A = ia$ for some $a \in \mathbb{R}$. Suppose $a \neq 0$. Set $x_0^{AA'} = a^{-1}\omega^A \overline{\omega}^{A'}$ which is Hermitian, so x_0^a is real. We now check

$$ix_0^{AA'}\pi_{A'} = ia^{-1}\omega^A\overline{\omega}^{A'}\pi_{A'} = (-ia)ia^{-1}\omega^A = \omega^A$$

as required. If $a = 0$ we change the origin of $\mathbb{C}M$ so that the incidence relation becomes

$$\omega^A = i(x^{AA'} - y^{AA'})\pi_{A'}$$

for some fixed $y^{AA'}$. The α-plane in M which was defined by Z^α is now defined by $\tilde{Z}^\alpha = (\tilde{\omega}^A, \pi_{A'})$ with $\tilde{\omega}^A = \omega^A + iy^{AA'}\pi_{A'}$. Choose $y^{AA'}$ such that

$$\tilde{\omega}^A\overline{\pi}_A = iy^{AA'}\pi_{A'}\overline{\pi}_A \neq 0$$

and we are done by the $a \neq 0$ case. □

Theorem 5.16. A null twistor corresponds to an α-plane whose real points define a null geodesic of M.

Proof. Let $Z^\alpha = (\omega^A, \pi_{A'})$ be a null twistor and $x_0^{AA'}$ be a real point on the corresponding α-plane. By Theorem 4.21

$$\{x_0^{AA'} + r\overline{\pi}^A\pi^{A'} \ : \ r \in \mathbb{R}\}$$

defines a null geodesic of M, which clearly lies within the α-plane defined by Z^α. Conversely suppose $x_1^{AA'}$ is another real point. Then

$$0 = (x_1^{AA'} - x_0^{AA'})\pi_{A'}$$

so for fixed A we have $(x_1^{AA'} - x_0^{AA'}) \propto \pi^{A'}$ whence $x_1^{AA'} - x_0^{AA'} = \lambda^A\pi^{A'}$ for some λ^A. Suppose $\{o^{A'}, \pi^{A'}\}$ form a dyad. We must have $\lambda^A\pi^{A'}$ Hermitian so

$$\lambda^A\pi^{A'} = \overline{\lambda}^{A'}\overline{\pi}^A$$

whence $\lambda^A = \overline{\lambda}^{A'}\overline{\pi}^A o_{A'}$. Contracting both sides with $\overline{\pi}_A$ shows $\lambda^A \propto \overline{\pi}^A$. □

Lemma 5.17. If $x_0^{AA'} \in M$ is a real point, then all corresponding twistors $Z^\alpha = (\omega^A, \pi_{A'})$ are null.

Proof. The incidence relation yields

$$\omega^A\overline{\pi}_A = ix_0^{AA'}\pi_A\pi_{A'}$$

and $(\star)$ immediately follows. □

Theorem 5.18. Two null geodesics in M meet iff their corresponding twistors X^α and Y^α satisfy $X^\alpha Y_\alpha = 0$.

Proof. This involves algebraic manipulations similar to those above, so we omit it. An argument may be found in Huggett and Tod [21, p. 56]. □

Corollary 5.19. The null cone at a point $p \in M$ corresponds to a projective line $L_p = \mathbb{P}^1 \subset \mathbb{P}N$.

Proof. Let X^α and Y^α be distinct twistors corresponding to fixed null geodesics through p. Suppose Z^α corresponds to an arbitrary null geodesic through p. By the previous theorem we must have

$$Z^\alpha Z_\alpha = 0\,, \quad Z^\alpha X_\alpha = 0\,, \quad Z^\alpha Y_\alpha = 0 \tag{$\dagger$}$$

Certainly these conditions are satisfied for all $Z^\alpha = \zeta X^\alpha + \eta Y^\alpha$ where $\zeta, \eta \in \mathbb{C}$. Moreover this is the general solution for Z^α since the second and third equations in (†) define a 2-dimensional complex subspace of T. Now projectifying the construction yields the result. □

Remark 5.20. Explicitly the line L_p thus constructed is nothing but

$$\{[\omega^A, \pi_{A'}] \in \mathbb{P}T : \omega^A = ip^{AA'}\pi_{A'}\}$$

as usual. The new information is that

$$\mathscr{C}^{-1}(L_p) \cap \mathbb{R} = \{\text{null cone at } p\}$$

Remark 5.21. We may summarise our geometrical findings as follows

$$\begin{array}{rcc}
M & \longleftrightarrow & \mathbb{P}N \\
\text{null cone at } p & \longleftrightarrow & \text{projective line } L_p = \mathbb{P}^1 \\
\text{null geodesic through } p & \longleftrightarrow & \text{point } [Z^\alpha] \text{ on } L_p \\
\begin{array}{c}\text{two points } p \text{ and } q \\ \text{are null separated}\end{array} & \longleftrightarrow & \begin{array}{c}\text{two lines } L_p \text{ and } L_q \\ \text{intersect at a point}\end{array}
\end{array}$$

We naturally interpret L_p as the celestial sphere of an observer at p.

Remark 5.22. Heuristically it might be convenient to regard twistor space as fundamental when quantising spacetime. Introducing small scale quantum behaviour in $\mathbb{P}T$ does not affect the null cone structure of M, which is determined by global lines of $\mathbb{P}T$. This ensures that causality is not violated. Instead the points of M themselves are subject to quantum uncertainty. This approach has particular merit in curved spacetimes, cf. Penrose and MacCallum [32].

Definition 5.23. A **null geodesic congruence** Γ through a region U of M is a set of null geodesics, one through each point of U.

Lemma 5.24. Suppose R^α is a twistor corresponding to a null geodesic $R \subset M$. Then R_α determines a null geodesic congruence through R.

Proof. Let $\Gamma = \mathscr{C}^{-1}(\{X^\alpha \in N : R_\alpha X^\alpha = 0\})$. Then Γ defines a null geodesic congruence through R by Theorem 5.18. $\square$

Remark 5.25. We now have a natural geometrical interpretation of dual null twistors. We may picture a generic dual twistor in $\mathbb{P}T^*$ by extending this argument. This provides the most tangible visual representation of a general twistor, and inspired the nomenclature.

Example 5.26. Given R_α in $\mathbb{P}T^*$ with $R^\alpha R_\alpha \neq 0$ we define the **Robinson congruence** of R_α by

$$\Gamma = \mathscr{C}^{-1}(\{X^\alpha \in \mathbb{P}T : R_\alpha X^\alpha = 0\})$$

Let $X^\alpha = (\omega^A, \pi_{A'}) \in \mathbb{P}T$ satisfy $R_\alpha X^\alpha = 0$. We are interested in visualising the locus X of real points on the α-plane defined by X^α. As usual we coordinatise X by (t, x, y, z) writing

$$x^{AA'} = \frac{1}{\sqrt{2}} \begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix}$$

Following Penrose [30] we describe a particular case of the general construction, noting that all Robinson congruences are related by Poincaré transformations. Write $R_\alpha = (A_A, A^{A'})$ then $(iA_A x^{AA'} + A^{A'})\pi_{A'} = 0$ whence

$$\pi^{A'} = k(iA_A x^{AA'} + A^{A'})$$

for some $k \in \mathbb{C}$. Let $0 \neq r \in \mathbb{R}$ and choose

$$R_\alpha = (0, \sqrt{2}, 0, -r)$$

so that $R_\alpha R^\alpha = -2r\sqrt{2} \neq 0$. Then in particular we have

$$\pi_{A'} = ik(x-iy,\, t-z+ir)$$

Now the tangent vector to X at (t, x, y, z) is given by

$$T^{AA'} = \bar{\pi}^A \pi^{A'} = |k|^2 \begin{pmatrix} x^2+y^2 & (x+iy)(t-z+ir) \\ (x-iy)(t-z-ir) & (t-z)^2+r^2 \end{pmatrix}$$

which corresponds to the vector

$$T^a = \frac{|k|^2}{\sqrt{2}} \begin{pmatrix} x^2 + y^2 + (t-z)^2 + r^2 \\ 2x(t-z) - 2yr \\ 2y(t-z) + 2rx \\ x^2 + y^2 - (t-z)^2 - r^2 \end{pmatrix}$$

The locus X is now given by the integral curves $x^a(s)$ of T^a in M, namely

$$\frac{dx^a}{ds} = T^a$$

For simplicity of visualisation we project X onto a hyperplane of constant $t = \tau$. Taking the parameter $s = (x^2+y^2+z^2)^{\frac{1}{2}}$ the reader may verify that the projected integral curves are given by

$$x^2 + y^2 + (\tau - z)^2 - 2r(x\cos\varphi - y\sin\varphi)\tan\theta = r^2 \qquad (\star)$$

$$(\tau - z) = (x\sin\varphi + y\cos\varphi)\tan\theta \qquad (\dagger)$$

for some real constants φ and θ. Eliminating φ and setting $\rho = x^2 + y^2$ we obtain

$$(\rho - r\sec\theta)^2 + (\tau - z)^2 = r^2\tan^2\theta \qquad (\diamond)$$

Now $(\star)$ describes a sphere and $(\dagger)$ a plane cutting the sphere. Thus the integral curves are circles. These circles, for varying φ but constant θ, lie on the surface defined by $(\diamond)$, which is a torus. For varying θ, we have a family of coaxial tori. The geometry is pictured in Penrose [31, §8].

Observe that the circles twist around the tori, each one linking with all the others. This immediately makes sense of the term 'twistor'. Interestingly, one can identify these circles with the fibres of the Hopf bundle over S^2. See, for example, Urbantke [37, §5].

Remark 5.27. Recall that every point in $\mathbb{C}M$ corresponds to a projective line $\mathbb{P}^1$ in $\mathbb{P}T$. We may gain an important new perspective on $\mathbb{C}M$ by geometrically classifying the set of all such lines in $\mathbb{P}T$. This will complete our analysis of the twistor correspondence.

Definition 5.28. A **quadric** in $\mathbb{P}^n$ is the projective variety defined by the vanishing of a quadratic form $Q(X) = a_{ij}X^iX^j$ in the homogeneous coordinates $\{X^0, \dots X^n\}$. A quadric is **non-degenerate** if Q is non-degenerate, or equivalently if a_{ij} is invertible.

Lemma 5.29. Let $F_{ab} \in \bigwedge^2 V$ with $\dim(V) = 4$. Then F_{ab} is simple iff

$$\varepsilon^{abcd}F_{ab}F_{cd} = 0$$

Proof. The forward direction is trivial. Conversely an elementary yet tedious calculation in components suffices. □

Example 5.30. Let L be the space of lines in $\mathbb{P}T$. An element $\ell \in L$ can be uniquely represented by two points $[X^\alpha]$ and $[Y^\beta]$ on ℓ. We may combine these to obtain a bivector $P^{\alpha\beta} = X^{[\alpha}Y^{\beta]}$. Clearly this is only defined up to scale for any given line. We therefore have an injection

$$\varphi : L \longrightarrow \mathbb{P}\left(\bigwedge^2 T\right) = \mathbb{P}^5$$

The image of φ is precisely the simple bivectors. By the previous lemma we may equivalently write

$$\mathrm{im}(\varphi) = \{[F_{ab}] \in \mathbb{P}^5 : \varepsilon^{abcd}F_{ab}F_{cd} = 0\}$$

Hence $\mathrm{im}(\varphi)$ defines a quadric in $\mathbb{P}^5$, called the **Klein quadric** Q. Under the twistor correspondence every $x \in \mathbb{C}M$ defines a line $\ell_x \in \mathbb{P}T$. Therefore we might hope to identify Q with $\mathbb{C}M$ in some way. To accomplish this rigorously we need to conformally compactify spacetime.

5.3 Conformal Compactification

In §4.4 we saw that Minkowski spacetime was conformally incomplete. It is therefore convenient to embed Minkowski space in a compact and conformally complete manifold before studying global theories. In this section we construct conformally compactified Minkowski space, and relate it to the Klein quadric of §5.2. We see that complexified conformally compactified Minkowski space is the natural arena for the twistor correspondence, motivating the formal approach followed in §5.4.

Definition 5.31. Let P be a 2-plane in a flat real manifold with metric g of indefinite signature. Then P has at most two independent null directions. We say that P is a **null plane** if it has exactly one null direction.

Remark 5.32. Note the subtle difference to Definition 5.9 of a *totally* null plane.

Lemma 5.33. Let $\mathbf{U}$ be null and $\mathbf{V}$ non-null in the null 2-plane P. Then $\mathbf{U}.\mathbf{V} = 0$. If $\mathbf{V}$ is spacelike then all non-null vectors in P are spacelike.

Proof. Note that $\mathbf{U}$ and $\mathbf{V}$ span P. Since $\mathbf{U}$ is the unique null direction we must have

$$0 \neq (a\mathbf{U} + b\mathbf{V})^2 = 2ab\mathbf{U}.\mathbf{V} + b^2\mathbf{V}^2$$

for all $a, b \in \mathbb{R}$ with $b \neq 0$. Therefore $\mathbf{U}.\mathbf{V} = 0$. If $\mathbf{V}$ is spacelike then a general non-null vector has norm squared

$$(a\mathbf{U} + b\mathbf{V})^2 = b^2\mathbf{V}^2 < 0$$

so is spacelike, as required. $\square$

Lemma 5.34. Let P and P' be null 2-planes with common null vector $\mathbf{U}$, each containing a spacelike vector. Define the angle θ between P and P' to be the angle between any non-null vectors, one in each plane. Then θ is well-defined.

Proof. Let $\mathbf{V}$ and $\mathbf{W}$ be distinguished spacelike vectors in P and P' respectively. Certainly the angle θ between $\mathbf{V}$ and $\mathbf{W}$ is well-defined. Let $a\mathbf{U}+b\mathbf{V}$, $c\mathbf{U}+d\mathbf{W}$ be general non-null vectors in each plane, with $b, d \neq 0$. Then these are spacelike by the previous lemma, so the angle $\tilde{\theta}$ between them is well-defined. We calculate

$$(a\mathbf{U} + b\mathbf{V}).(c\mathbf{U} + d\mathbf{W}) = bd\mathbf{V}.\mathbf{W}$$

and note that $(a\mathbf{U} + b\mathbf{V}) = b^2\mathbf{V}^2$, $(a\mathbf{U} + d\mathbf{W}) = d^2\mathbf{W}^2$. Now it is immediate that $\cos\theta = \cos\tilde{\theta}$. $\square$

Construction 5.35. Let E be a real six dimensional manifold with a preferred coordinate chart

$$A = (T, V, W, X, Y, Z)$$

and metric components

$$g = \mathrm{diag}(+1, +1, -1, -1, -1, -1)$$

Let N be the light cone of the origin, in other words

$$N = \{A \in E : g(A, A) = 0\}$$

Since N is defined by a homogeneous polynomial we may projectify to obtain $\mathbb{P}N \subset \mathbb{P}^5$, a closed subset of a compact space, so compact. Moreover $\mathbb{P}N$ is four-dimensional. Note that $\mathbb{P}N$ is a quadric in the sense of Definition 5.28.

The metric on E induces a conformal metric on $\mathbb{P}N$. Indeed we may choose representatives for $\mathbb{P}N$ by intersecting N with any spacelike hypersurface not through 0. Let S and S' be two such hypersurfaces cutting the future null cone N^+. We show that the induced metrics on $S \cap N$ and $S' \cap N$ are conformally equivalent.

We follow a beautiful argument of Penrose and Rindler [33, p. 38]. Consider three infinitesimally close lines in N, say ℓ_i for $i = 1, 2, 3$. Let s_i be an arbitrary point on ℓ_i in the future null cone N^+. Let s_{ij} be the infinitesimal spacelike lines defined by the pairs $\{s_i, s_j\}$ of distinct points. Let P_{ij} be the null 2-planes through 0 containing s_{ij}. Note that P_{ij} depend only on the lines ℓ_i not on the choice of points s_i.

Now let s_i and s_i' be defined by the intersection of S and S' with ℓ_i. Then $\{s_i\}$ and $\{s_i'\}$ define infinitesimal triangles on N, whose angles are determined by the induced metrics on $S \cap N$ and $S' \cap N$ respectively. If these triangles are similar then we may conclude that the metrics are conformally equivalent. But this is immediate from Lemma 5.34 and our arguments in the previous paragraph.

Let $\tilde{g}$ be the induced conformal metric on $\mathbb{P}N$. Then we call $(\mathbb{P}N, \tilde{g})$ **compactified Minkowski space**, and denote it M^c.

Theorem 5.36. Define a smooth map $\varphi : M \longrightarrow E$ by

$$X^a \longmapsto (X^0, \frac{1}{2}(1 - X^b X_b), -\frac{1}{2}(1 + X^b X_b), X^1, X^2, X^3)$$

Then $\mathrm{im}(\varphi) = N \cap Z$ where $Z = \{A \in E : V - W = 1\}$. Hence φ defines an isometric embedding of M into N.

Proof. Trivially we check

$$\varphi(X^a)^2 = X^b X_b - \frac{1}{4}(1 - X^b X_b)^2 + \frac{1}{4}(1 + X^b X_b)^2 = X^b X_b - X^b X_b$$

so $\mathrm{im}(\varphi) \subset N$. Also by definition $\mathrm{im}(\varphi) \subset Z$. For surjectivity, suppose

$$(T, V, W, X, Y, Z) \in Z \cap N$$

and $X^a = (T, X, Y, Z)$. Then

$$\frac{1}{2}(1 - X^b X_b) = \frac{1}{2}(1 - T^2 + X^2 + Y^2 + Z^2) = \frac{1}{2}(1 - W^2 + V^2) = V$$

and similarly for W. Injectivity is obvious, and isometry follows since $dV = dW$ on Z, so g reduces to the Minkowski metric. $\square$

Corollary 5.37. As a set M^c is Minkowski space together with an extra null cone and 2-sphere.

Proof. Suppose ℓ is a null line generating N, that is $\ell \in \mathbb{P}N$. If $V - W \neq 0$ on ℓ then there is exactly one point in $\ell \cap Z$. Hence M conformally embeds in $\mathbb{P}N$.

Consider the remaining points of $\mathbb{P}N$, precisely the set $\mathbb{P}K$ for $K = \{A \in N : V - W = 0\}$. Then

$$\mathbb{P}K = \mathbb{P}S \cup \mathbb{P}C$$

where $C = \{A \in N : V = W \neq 0\}$ and $S = \{A \in N : V = W = 0\}$.

Every element of $\mathbb{P}C$ may be uniquely represented by a point $A \in N$ with $V = W = 1$. Therefore

$$\mathbb{P}C = \{(T, X, Y, Z) : T^2 - X^2 - Y^2 - Z^2 = 0\}$$

has the structure of the null cone in M.

In homogeneous coordinates we may write

$$\mathbb{P}S = \{[T : X : Y : Z] : T^2 - X^2 - Y^2 - Z^2 = 0\}$$

Clearly we must have $T \neq 0$. Thus every element of $\mathbb{P}S$ may be uniquely represented by a point $A \in N$ with $V = W = 0$ and $T = 1$. This identifies $\mathbb{P}S$ with S^2. □

Remark 5.38. Huggett and Tod [21, p. 36], fail to mention the extra copy of S^2, as noted in Jadcyzk [24].

Theorem 5.39. *M^c is a conformal completion of M.*

Proof. By Liouville (Theorem 4.61) it suffices to exhibit a global conformal transformation of M^c which has the form of an inversion when restricted to $M \setminus \{\text{null cone of } 0\}$. Let $f : \mathbb{P}N \longrightarrow \mathbb{P}N$ be a reflection in the plane $V = 0$. Clearly this is a conformal transformation of M^c. Moreover taking $\Delta = X^b X_b$ and using homogeneous coordinates we have

$$\begin{aligned} f \circ \varphi(X^a) &= [X^0 : -\frac{1}{2}(1 - \Delta) : -\frac{1}{2}(1 + \Delta) : X^1 : X^2 : X^3] \\ &= [X^0/\Delta : \frac{1}{2}(1 - 1/\Delta) : -\frac{1}{2}(1 + 1/\Delta) : X^1/\Delta : X^2/\Delta : X^3/\Delta] \\ &= \varphi \circ g(X^a) \end{aligned}$$

where $g(X^a) = X^a/(X^b X_b)$ is the inversion map. □

Remark 5.40. Note that f maps the null cone at 0 to the extra null cone $\mathbb{P}C$. For this reason we often regard $\mathbb{P}C$ is being at infinity. Alternative approaches to compactification identify $\mathbb{P}K$ with the boundary of Minkowski space, so it is natural to place $\mathbb{P}S$ at infinity also. See, for example, Penrose and Rindler [34].

Definition 5.41. Complexified compactified Minkowski space is the complexified space $\mathbb{CP}N$ with the induced metric from $(\mathbb{C}E, g)$. We denote it by $\mathbb{C}M^c$ or M.

Theorem 5.42. Complexified compactified Minkowski space is diffeomorphic to the Klein quadric.

Proof. Let $\bigwedge^2 T$ be the 6-dimensional space of bivectors of T, with coordinates $P^{\alpha\beta}$. Recall that the Klein quadric is the subspace of $\mathbb{P}^5$ defined by the homogeneous equation

$$P^{12}P^{34} - P^{13}P^{24} + P^{14}P^{23} = 0 \tag{$\star$}$$

It suffices to exhibit coordinates (T, V, W, X, Y, Z) for $\bigwedge^2 T$ such that $(\star)$ becomes

$$T^2 + V^2 - W^2 - X^2 - Y^2 - Z^2 = 0$$

Indeed we may choose $P^{12} = (T + X)$, $P^{34} = (T - X)$, $P^{14} = (V + W)$, $P^{23} = (V - W)$, $P^{13} = (Y + iZ)$, $P^{24} = (Y - iZ)$ and the proof is complete. □

Remark 5.43. Recall that in §5.2 we identified that Klein quadric with the space of lines in $\mathbb{P}T$, which is precisely the Grassmanian of 2-planes in T. We may hence regard M as this Grassman manifold. This provides a more abstract perspective on the twistor correspondence, which we now articulate.

5.4 The Formal Twistor Correspondence

Definition 5.44. We define the **Grassmanian** of k-planes in $\mathbb{C}^n$ by

$$\mathbb{G}_{n,k} = \{k\text{-dimensional subspaces of } \mathbb{C}^n\}$$

Lemma 5.45. $\mathbb{G}_{n,k}$ is a manifold of dimension $(n - k) \times k$.

Proof. Let $\mathbb{C}_*^{n\times k}$ be the set of $(n \times k)$ matrices of maximal rank. Define a mapping

$$[\quad] : \mathbb{C}_*^{n\times k} \longrightarrow \mathbb{G}_{n,k}$$

by taking $[m]$ to be the span of the columns of m. We think of m as a homogeneous coordinate for $\mathbb{G}_{k,n}$, generalising the case $\mathbb{G}_{1,n} = \mathbb{P}^n$. Define a coordinate chart $\varphi_1 : \mathbb{C}^{(n-k)\times k} \longrightarrow \mathbb{G}_{n,k}$ by

$$\varphi_1(Z) = \left[\begin{pmatrix} iZ \\ I_k \end{pmatrix} \right]$$

where I_k denotes the $(k \times k)$ identity matrix. Simple topological considerations demonstrate that φ_1 is a homeomorphism. We obtain the remaining coordinate charts combinatorically, by permuting the rows in the image of φ_1. This operation can be realised as the action of an element of $GL(n, \mathbb{C})$, so the transition functions are biholomorphic. Thus we have endowed $\mathbb{G}_{n,k}$ with a manifold structure. □

Remark 5.46. Recall from §5.3 that we may regard M as the Grassmanian of

2-planes in T. We introduce the notation $\mathsf{M}^I = \varphi_1(\mathbb{C}^{2\times 2}) \subset \mathsf{M}$ and wlog identify M^I with $\mathbb{C}M$. Symbolically M^I represents M with the points I at infinity removed. Explicitly we write

$$\varphi_1(x^{AA'}) = \left[\begin{pmatrix} ix^{AA'} \\ I_2 \end{pmatrix}\right]$$

Definition 5.47. View M as the Grassmanian of 2-planes in T, and write $\mathsf{P} = \mathbb{P}T$. Define the **correspondence space** by

$$\mathsf{F} = \{(V_1, V_2) : V_i \text{ subspaces of } T \text{ of dimension } i \text{ and } V_1 \subset V_2\}$$

There are natural projection maps $\mu : \mathsf{F} \longrightarrow \mathsf{P}$ and $\nu : \mathsf{F} \longrightarrow \mathsf{M}$ defining the **double fibration**

$$\begin{tikzcd} & \mathsf{F} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ \mathsf{P} & & \mathsf{M} \end{tikzcd}$$

We define the **twistor correspondence** $\mathscr{C} : \mathsf{M} \longrightarrow \mathsf{P}$ by $\mathscr{C} = \mu \circ \nu^{-1}$. We denote $\nu^{-1}(\mathsf{M}^I) = \mathsf{F}^I$ and $\mathscr{C}^{-1}(\mathsf{M}^I) = \mathsf{P}^I$.

Theorem 5.48. We may endow F with the structure of the projective dual primed spin bundle $\mathbb{P}\mathcal{S}'^*$ over M.

Proof. We exhibit a local trivialisation

$$\psi : \mathsf{M}^I \times \mathbb{P}^1 \longrightarrow \mathsf{F}^I$$

Taking coordinates $x^{AA'}$ on M^I and $[\pi_{A'}]$ on $\mathbb{P}^1$ we define

$$\psi(x^{AA'}, \pi_{A'}) = \left(\begin{bmatrix} ix^{AA'}\pi_{A'} \\ \pi_{A'} \end{bmatrix}, \begin{bmatrix} ix^{AA'} \\ I_2 \end{bmatrix} \right)$$

Now extending Lemma 5.45 we may regard F as a manifold and verify that ψ has the appropriate properties. The details are messy and unimportant, so we leave them to the diligent reader. Finally we construct the remaining trivialisations using the combinatorial arguments of Lemma 5.45. $\square$

Remark 5.49. In coordinates $[\omega^A, \pi_{A'}]$ on P^I and $x^{AA'}$ on M^I the double fibration may explicitly be written

$$\begin{tikzcd}[column sep=tiny] & \left(\begin{bmatrix} ix^{AA'}\pi_{A'} \\ \pi_{A'} \end{bmatrix}, \begin{bmatrix} ix^{AA'} \\ I_2 \end{bmatrix}\right) \arrow[dl] \arrow[dr] & \\ \left[ix^{AA'}\pi_{A'}, \pi_{A'}\right] & & x^{AA'} \end{tikzcd}$$

This demonstrates that the twistor correspondence of Definition 5.47 is a formal generalisation of Definition 5.8, as we might hope.

Remark 5.50. Defining $F = \mathbb{C}M \times (S'^{*} \setminus \{0\})$ we obtain a double fibration

$$\begin{tikzcd} & F \arrow[dl] \arrow[dr] & \\ T \setminus \{0\} & & \mathbb{C}M \end{tikzcd}$$

given explicitly by the formulae in the previous remark, without projectification. Although this perspective is only valid locally, it suffices for many calculations. Moreover it is notationally and conceptually easier than our earlier version. We employ this approach frequently in §6.

Remark 5.51. We define F^+ and F^- analogously to F^+ and F^-.

6 Twistor Transforms

In this final section we draw together the disparate threads explored above. Twistor transforms admit study from a variety of different perspectives, of which we introduce the most intuitive. Broadly speaking, a twistor transform relates fields defined on Minkowski space with functions or bundles on twistor space. By choosing our language correctly we can promote this relationship to a bijection. We then have the freedom to translate problems between these two perpectives to find novel means of solution.

We begin by considering the integral formulae mentioned in §1.1. We see that twistor functions naturally encode solutions to the ZRM equations on Minkowski space, a striking result. Motivated by a desire to invert the Penrose transform, we turn to the power of sheaf cohomology. A precise interpretation swiftly emerges.

We conclude the section by exploring a nonlinear generalization of the Penrose transform due to Ward [38]. We see that the anti-self-dual solutions of the Yang-Mills equations can be classified in terms of vector bundles on twistor space. The ASD condition on a gauge field is naturally expressed as a compatibility condition for overlapping trivialisations. This philosophy has natural applications to theories of instantons and monopoles, which we allude to in §7.

6.1 Integral Formulae

Definition 6.1. A **twistor function** is a function $f(Z^\alpha)$ on twistor space.

Definition 6.2. We define the **future tube** of complexified Minkowski space by

$$\mathbb{C}M^+ = \mathscr{C}^{-1}(T^+)$$

Remark 6.3. Recall that in quantum field theory we discard negative frequency fields, for they correspond to unphysical negative energy particles. Therefore we are most interested in solving the ZRM equations for positive frequency fields. Following Hughston and Ward [22, p. 21] we note without proof that a field $\varphi_{A\dots B}$ on Minkowski space is of positive frequency if it can be extended to the forward tube $\mathbb{C}M^+$ by analytic continuation. Using hyperfunctions one may obtain the converse statement also, cf. Bailey et al. [3]. Motivated by this, we shall seek solutions of the ZRM equations defined on $\mathbb{C}M^+$.

Theorem 6.4. Recall the helicity $n/2$ ZRM equations for a valence n spinor field $\varphi_{A'\dots B'}$, namely

$$\nabla^{AA'}\varphi_{A'\dots B'} = 0$$

These have solutions on $\mathbb{C}M^+$ given by

$$\varphi_{A'\ldots B'}(x) = \frac{1}{2\pi i}\oint \pi_{A'}\ldots\pi_{B'}\rho_x f(Z^\alpha)\pi_{C'}d\pi^{C'}$$

where

- f is homogeneous of degree $(-n-2)$ in Z^α
- $Z^\alpha = (\omega^A, \pi_{A'})$
- ρ_x denotes restriction to the line $\mathbb{P}^1 \subset \mathbb{P}T$ defined by x via the twistor correspondence
- $\pi_{A'}$ are homogeneous coordinates on $\mathbb{P}^1$
- the contour is arbitrary, provided it avoids the singularities of f and varies continuously with x

Proof. First observe that the integral is well-defined on $\mathbb{P}^1$, since the entire integrand (including the differential) has homogeneity 0 in $\pi_{A'}$. Applying the chain rule we obtain

$$\nabla_{AA'}\rho_x f(Z^\alpha) = \frac{\partial}{\partial x^{AA'}}\rho_x f(\omega^A, \pi_{A'}) = \rho_x \frac{\partial f}{\partial \omega^C}\frac{\partial \omega^C}{\partial x^{AA'}} = i\pi_{A'}\rho_x \frac{\partial f}{\partial \omega^A}$$

Now differentiating under the integral sign we get

$$\nabla_{CC'}\varphi_{A'\ldots B'} = \frac{1}{2\pi}\oint \pi_{A'}\ldots\pi_{B'}\pi_{C'}\rho_x \frac{\partial f}{\partial \omega^C}\pi_{E'}d\pi^{E'}$$

which is clearly symmetric in $A'\ldots C'$ and so satisfies the ZRM equations in the form of Lemma 4.50. ☐

Remark 6.5. By Remark 3.39 we may regard f as a section of $\mathcal{O}(-n-2)$ on $\mathbb{P}^3$. We adopt this viewpoint more explicitly in §6.2.

Remark 6.6. Our proof is incomplete, for we have not demonstrated that an appropriate contour exists. We see in Example 6.8 that this is indeed a nontrivial problem. We leave this subtle point to the rigorous methods of §6.2. There, we solve the problem using the fact that $\mathbb{C}M^+$ is Stein.

Theorem 6.7. The helicity $-n/2$ ZRM equations for a valence n spinor field $\varphi_{A\ldots B}$ have solutions on $\mathbb{C}M^+$ given by

$$\varphi_{A\ldots B}(x) = \frac{1}{2\pi i}\oint \rho_x \frac{\partial}{\partial \omega^A}\ldots\frac{\partial}{\partial \omega^B} f(Z^\alpha)\pi_{C'}d\pi^{C'}$$

where f is homogeneous of degree $(n-2)$ in Z^α and all other notation is as in the previous theorem.

Proof. Trivial from the previous proof. □

Example 6.8 (Wave equation). The alert reader may notice that we have not explicitly verified our formulae in the case $n = 0$. This is not hard to check, so instead we compute an example to develop our intuition. Consider the twistor function

$$f(Z^\alpha) = \frac{1}{(A_\alpha Z^\alpha)(B_\beta Z^\beta)}$$

This has homogeneity -2 in Z^α so applying Theorem 6.4 should yield a solution to the wave equation. For convenience set

$$\alpha^{A'} = iA_A x^{AA'} + A^{A'} \quad \text{and} \quad \beta^{A'} = iB_A x^{AA'} + B^{A'}$$

so that the integral reads

$$\varphi(x) = \frac{1}{2\pi i} \oint \frac{1}{(\alpha^{A'} \pi_{A'})(\beta^{B'} \pi_{B'})} \pi_{C'} d\pi^{C'}$$

Observe that an appropriate contour exists iff the poles are distinct. Indeed any choice of contour varying continuously with x and enclosing one of the poles becomes singular when the poles coincide. If we want $\varphi(x)$ to be well-defined on $\mathbb{C}M^+$ we need to place some restriction on A_α and B_β.

Now A_α and B_α define a line L in $\mathbb{P}T$ and hence a point $y \in \mathsf{M}$ via the dual twistor correspondence. By a complex extension of Theorem 5.18 we see that $\varphi(x)$ is singular at precisely those $x \in \mathbb{C}M$ which are complex null separated from y. Appealing to a complexified version of Remark 5.21 we have that $\varphi(x)$ is singular iff $L_x \equiv \mathscr{C}(x)$ intersects L in $\mathbb{P}T$. Therefore it suffices to choose A_α and B_α such that L lies entirely in $\mathbb{P}T^-$ for φ to be well-defined on $\mathbb{C}M^+$.

We may now assume that the poles are distinct, so in particular $\alpha^{A'} \beta_{B'} \neq 0$. Let z be a coordinate on $\mathbb{P}^1$ given by

$$\pi_{A'} = \alpha_{A'} + z\beta_{A'}$$

Then the integral becomes

$$\varphi(x) = \frac{1}{2\pi i} \oint \frac{dz}{(\alpha^{A'} \beta_{A'})z} = \frac{1}{\alpha^{A'} \beta_{A'}}$$

by the residue theorem. Now since A_α and B_β lie on the line defined by y we have, by the dual twistor correspondence

$$A^{A'} = -iy^{AA'} A_A \quad \text{and} \quad B^{A'} = -iy^{AA'} B_A$$

whence we obtain

$$\alpha^{A'}\beta_{A'} = A_A x^{AA'} B^B x_{BA'} - A_A y^{AA'} B^B x_{BA'} \\ - A_A x^{AA'} B^B y_{BA'} + A_A y^{AA'} B^B y_{BA'}$$

Now using the relations

$$x^{0A'}x_{1A'} = x^{00'}x_{10'} + x^{01'}x_{11'} = x_{11'}x_{10'} - x_{10'}x_{11'} = 0 \\ x^{0A'}x_{0A'} = x^{1A'}x_{1A'}$$

we may conclude that

$$A_A B^B x^{AA'} x_{BA'} = \frac{1}{2} A_A B^A x^2$$

Treating the other terms similarly we obtain

$$\varphi(x) = \frac{2}{A_A B^A (x-y)^2}$$

It is now trivial to check that $\varphi(x)$ satisfies the wave equation, as required.

Example 6.9 (ASD Coulomb field). In Hughston and Ward [22, p. 137] it is claimed that the twistor function

$$f(Z^\alpha) = \log \frac{Z^1 Z^2 - Z^0 Z^3}{Z^2 Z^3}$$

produces an ASD Coulomb field $F^{\mu\nu}$ where $F^{0j} \equiv E^j \equiv iB^j$ and

$$\mathbf{E} \propto \mathbf{r}/r^3$$

Let F be an ASD Coulomb field. Then by Theorem 4.30 we may write

$$F_{ab} = F_{AA'BB'} = \varphi_{AB}\epsilon_{A'B'}$$

In particular we have

$$E_x = F_{01} = -\varphi_{01} \\ E_y = F_{02} = \frac{1}{2}(\varphi_{11} - \varphi_{00}) \\ E_z = F_{03} = -\frac{1}{2}i(\varphi_{00} + \varphi_{11})$$

Now we calculate φ_{AB} using the contour integral formula

$$\begin{aligned}\varphi_{AB}(t,x,y,z) &= \frac{1}{2\pi i}\oint \rho_x \frac{\partial}{\partial\omega^A}\frac{\partial}{\partial\omega^B} f(Z^\alpha)\pi_{E'}d\pi^{E'} \\ &= \frac{1}{2\pi i}\oint \frac{(\delta_A^0\pi_{1'} - \delta_A^1\pi_{0'})(\delta_B^0\pi_{1'} - \delta_B^1\pi_{1'})}{(x^{1A'}\pi_{A'}\pi_{0'} - x^{0A'}\pi_{A'}\pi_{1'})^2}\pi_{E'}d\pi^{E'}\end{aligned}$$

Choosing local coordinates $\pi_{E'} = (1,\xi)$ and using the convention

$$\begin{pmatrix} x^{00'} & x^{01'} \\ x^{10'} & x^{11'} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} t+x & y+iz \\ y-iz & t-x \end{pmatrix}$$

we get

$$\varphi_{AB} = \frac{1}{2\pi i}\oint d\xi \frac{(\delta_A^1 - \delta_A^0\xi)(\delta_B^1 - \delta_B^0\xi)}{(1/\sqrt{2}(y-iz) + \sqrt{2}x\xi - 1/\sqrt{2}(y+iz)\xi^2)^2}$$

This has double poles at

$$\xi = \frac{-\sqrt{2}x \pm \sqrt{2x^2+2y^2+2z^2}}{-\sqrt{2}(y+iz)} = \frac{x \mp r}{y+iz}$$

Denote these ξ_1 and ξ_2. The residue at ξ_1 is

$$\begin{aligned} r_1 &= \rho_{\xi_1}\frac{d}{d\xi}\frac{2(\delta_A^1 - \delta_A^0\xi)(\delta_B^1 - \delta_B^0\xi)}{(y+iz)^2(\xi-\xi_2)^2} \\ &= \frac{1}{2r^2}\Big(-\delta_A^0(\delta_B^1 - \delta_B^0\xi_1) - \delta_B^0(\delta_A^1 - \delta_A^0\xi_1) \\ &\qquad + (\delta_A^1 - \delta_A^0\xi_1)(\delta_B^1 - \delta_B^0\xi_1)(y+iz)/r\Big)\end{aligned}$$

Now we calculate explicitly

$$\begin{aligned}\varphi_{01} &= \frac{1}{2r^2}(-1 - \xi_1(y+iz)/r) = -\frac{x}{2r^3} \\ \varphi_{00} &= \frac{1}{2r^2}(2\xi_1 + \xi_1^2(y+iz)/r) = -\frac{(y-iz)}{2r^3} \\ \varphi_{11} &= \frac{(y+iz)}{2r^3}\end{aligned}$$

whence we find

$$E_x = \frac{x}{2r^3}, \quad E_y = \frac{y}{2r^3}, \quad E_z = \frac{z}{2r^3}$$

as required.

Remark 6.10. It is natural to ask whether we can formulate an inverse twistor transform. Given a ZRM field φ on $\mathbb{C}M^+$, what is the set of twistor functions which yield φ under the Penrose integral? This is not immediately obvious. Suppose we are given f producing φ via the integral formula with contour Γ at x. Let h and $\tilde{h}$ be holomorphic on opposite sides of Γ. Then certainly $f + h - \tilde{h}$ will also generate φ. This freedom should remind the reader of our discussion of sheaf cohomology in §2.3. Indeed we now proceed to reformulate the ideas of this section in the language of sheaves, thus obtaining a bijective transform.

6.2 The Penrose Transform

Lemma 6.11. A function $f(x^{AA'}, \pi^{A'})$ on F pushes down to a function on P iff $\pi^{A'}\nabla_{AA'}f = 0$ in every coordinate chart.

Proof. We demonstrate that this is equivalent to the stated condition in our preferred patch $(\mathsf{P}^I, \mathsf{M}^I, \mathsf{F}^I)$. Then the general result follows by a combinatorial argument. Clearly $f(x^{AA'}, \pi_{A'})$ yields a function on P^I iff it is constant each α-plane defined by $x^{AA'}$ and $\pi_{A'}$. We observe

$$\begin{aligned} \pi^{A'}\nabla_{AA'}f = 0 \quad &\Leftrightarrow \quad \nabla_{AA'}f = \xi_A \pi_{A'} \text{ for some } \xi_A(\pi) \\ &\Leftrightarrow \quad f = \xi_A \pi_{A'} x^{AA'} = \xi_A \omega^A \end{aligned}$$

and the result follows. □

Remark 6.12. In particular a function $f(x^{AA'}, \pi_{A'})$ on F pushes down to a twistor function iff the given condition holds in the non-projective sense. We shall make frequent use of this observation.

Theorem 6.13.

$$H^1(\mathbb{P}T^+, \mathcal{O}(-n-2)) \cong \{\text{ZRM fields } \varphi_{A'\ldots B'} \text{ of helicity } n/2 \text{ on } \mathbb{C}M^+\}$$

where we may view the set of ZRM fields as a group under addition since the ZRM equations are linear.

Proof. The flavour of the proof is as follows. We construct a short exact sequence of sheaves culminating in the sheaf of germs of the desired ZRM fields. Recalling from §2.3 the long exact sequence in cohomology, we obtain the required isomorphism by identifying certain sheaves as zero.

Define the sheaves $\mathcal{Z}_n(m)$ on F^+ by stipulating that $\varphi_{A'\ldots B'}(x, \pi) \in \mathcal{Z}_n(m)$ must satisfy the following conditions

- $\varphi_{A'\ldots B'}$ is a symmetric holomorphic valence n primed spinor field on F^+
- $\varphi_{A'\ldots B'}$ is homogeneous of degree m in π

- $\varphi_{A'\ldots B'}$ satisfies the ZRM equation $\nabla^{AA'}\varphi_{A'\ldots B'}$ throughout F^+

Note immediately that $\mathcal{Z}_n(0)$ consists of symmetric n index primed spinor fields which are independent of π, so there is a canonical sheaf isomorphism

$$\mathcal{Z}_n(0) \cong \{\text{ZRM fields } \varphi_{A'\ldots B'} \text{ of helicity } n/2 \text{ on } \mathbb{C}M^+\}$$

Define a sheaf morphism

$$\begin{aligned} P : \mathcal{Z}_{n+1}(m-1) &\longrightarrow \mathcal{Z}_n(m) \\ \varphi_{A'B'\ldots C'} &\longmapsto \pi^{A'}\varphi_{A'B'\ldots C'} \end{aligned}$$

We claim that this morphism is surjective, and it suffices to check this locally by Theorem 2.25. Let $\psi_{B'\ldots C'} \in \mathcal{Z}_n(m)$ be arbitrary. Define pointwise for each $(x^{AA'}, \pi_{A'}) \in F^+$

$$\begin{aligned} \varphi_{0B'\ldots C'} &= \frac{1}{2\pi^0}\psi_{B'\ldots C'} \\ \varphi_{1B'\ldots C'} &= \frac{1}{2\pi^1}\psi_{B'\ldots C'} \end{aligned}$$

which we can do since $\pi_{A'} \neq 0 \in F$ by definition. When $\pi_{0'} = 0$ or $\pi_{0'} = 0$ individually an obvious modification can be made. Then clearly $\varphi_{A'\ldots C'} \in \mathcal{Z}_{n+1}(m-1)$ and around every point of F^+ there exists a neighbourhood in which $P(\varphi_{A'\ldots C'}) = \psi_{B'\ldots C'}$.

Consider the special case $m = 0$. Let K denote the sheaf kernel of $P : \mathcal{Z}_{n+1}(-1) \longrightarrow \mathcal{Z}_n(0)$. Define on F^+ the sheaves

$$\begin{aligned} \mathcal{T}(n) = &\{\text{scalar fields } f(x,\pi) \text{ homogeneous of degree } n \\ &\text{in } \pi \text{ which push down to twistor functions}\} \end{aligned}$$

We claim that K is isomorphic to $\mathcal{T}(-n-2)$. Indeed let $\chi_{A'\ldots B'} \in K$ be an $(n+1)$ index spinor field on F^+, homogeneous of degree -1 in π. Then since $\chi_{A'\ldots B'}$ symmetric we may write

$$\chi_{A'\ldots B'} = \alpha_{(A'} \ldots \beta_{B')}$$

using Lemma 4.31. We then deduce

$$\begin{aligned} \pi^{A'}\alpha_{(A'} \ldots \beta_{B')} = 0 \quad &\Rightarrow \quad \pi^{A'} \ldots \pi^{B'}\alpha_{(A'} \ldots \beta_{B')} = 0 \qquad (\dagger) \\ &\Rightarrow \quad \pi^{A'}\alpha_{A'} \ldots \pi^{B'}\beta_{B'} = 0 \end{aligned}$$

$$\begin{aligned}
&\overset{\text{wlog}}{\Rightarrow} \quad \pi^{A'}\alpha_{A'} = 0 \\
&\Rightarrow \quad \pi^{A'}\alpha_{A'} = 0, \ldots \pi^{B'}\beta_{B'} = 0 \text{ by } (\dagger) \text{ and induction} \\
&\Rightarrow \quad \chi_{A'\ldots B'} = \pi_{A'}\ldots\pi_{B'}f(x,\pi)
\end{aligned}$$

Now since $\pi \neq 0$ the ZRM equations imply

$$\pi_{A'}\nabla^{AA'}f = 0$$

which is precisely the condition that f pushes down to a twistor function. Observe also that f is homogeneous of degree $(-n-2)$ in π. The converse is obvious.

We thus have a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{T}(-n-2) \xrightarrow{\pi_{A'}\ldots\pi_{B'}} \mathcal{Z}_{n+1}(-1) \xrightarrow{\pi^{A'}} \mathcal{Z}_n(0) \longrightarrow 0$$

whence we obtain a long exact sequence of cohomology

$$\begin{aligned}
\ldots \longrightarrow H^0(F^+, \mathcal{Z}_{n+1}(-1)) \longrightarrow H^0(F^+, \mathcal{Z}_n(0)) \xrightarrow{\delta^*} \\
H^1(F^+, \mathcal{T}(-n-2)) \longrightarrow H^1(F^+, \mathcal{Z}_{n+1}(-1)) \longrightarrow \ldots
\end{aligned}$$

We now identify these groups.

- Suppose $s(x,\pi) \in H^0(F^+, \mathcal{Z}_{n+1}(-1))$. Then s is a global section of $\mathcal{Z}_{n+1}(-1)$ over F^+. For fixed x, s defines a global section of $\mathcal{O}(-1)$ over $\mathbb{P}^1$, so $s = 0$ by Lemma 3.37. Thus $H^0(F^+, \mathcal{Z}_{n+1}(-1)) = 0$.
- $H^0(F^+, \mathcal{Z}_n(0))$ is clearly the desired group of ZRM fields on F^+.
- Observe that we may canonically identify $\mathcal{T}(-n-2)$ with the sheaf of twistor functions homogeneous of degree $(-n-2)$ on T^+, which itself is naturally intepreted as the sheaf $\mathcal{O}(-n-2)$ on $\mathbb{P}T^+$. We may therefore write $H^1(F^+, \mathcal{T}(-n-2)) \cong H^1(\mathbb{P}T^+, \mathcal{O}(-n-2))$.
- Following Hughston and Ward [22, p. 61] we note without proof that $\mathbb{C}M^+$ is Stein. Since $\mathcal{Z}_{n+1}(-1)$ is a sheaf of holomorphic sections of a vector bundle it is coherent analytic by Remark 2.42. Thus the pullback $\mathcal{G}$ of $\mathcal{Z}_{n+1}(-1)$ to $\mathbb{C}M^+$ has $H^1(\mathbb{C}M^+, \mathcal{G}) = 0$. Recall from Theorem 3.42 that $H^1(\mathbb{P}^1, \mathcal{O}(-1)) = 0$. Hence the pullback $\mathcal{H}$ of $\mathcal{Z}_{n+1}(-1)$ to $\mathbb{P}^1$ has $H^1(\mathbb{P}^1, \mathcal{H}) = 0$. Applying a suitable Künneth formula, cf. Sampson and Washnitzer [35], we get $H^1(F^+, \mathcal{Z}_{n+1}(-1)) = 0$.

Therefore we may conclude that δ^* provides the required isomorphism in the statement of the theorem, and our proof is complete. $\square$

Remark 6.14. We may regain the contour integral formulation of the Penrose transform by explicitly analysing the map $(\delta^*)^{-1}$. Recall that to define δ^* we consider the cochain complex of sheaves on F^+

$$\begin{tikzcd}
0 \arrow[r] & C^0(\mathcal{T}(-n-2)) \arrow[r] \arrow[d, "d"] & C^0(\mathcal{Z}_{n+1}(-1)) \arrow[r] \arrow[d, "d"] & C^0(\mathcal{Z}_n(0)) \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & C^1(\mathcal{T}(-n-2)) \arrow[r] \arrow[d, "d"] & C^1(\mathcal{Z}_{n+1}(-1)) \arrow[r] \arrow[d, "d"] & C^1(\mathcal{Z}_n(0)) \arrow[r] \arrow[d, "d"] & 0 \\
& {} & {} & {} &
\end{tikzcd}$$

We reverse the steps in Theorem 2.48 to determine $(\delta^*)^{-1}$. Choose a cover which is Leray for all the given sheaves on F^+ and work with Čech cohomology. Let $f_{ij} \in H^1(\mathbb{P}T, \mathcal{O}(-n-2))$. Then by commutativity of the above diagram

$$\pi_{A'} \dots \pi_{C'} f_{ij} \in H^1(\mathcal{Z}_{n+1}(-1)) = 0$$

Therefore we may write

$$\pi_{A'} \dots \pi_{C'} f_{ij} = \rho_{[i} \psi_{j]\, A' \dots C'}$$

for some $\psi_{j\, A' \dots C'} \in C^0(\mathcal{Z}_{n+1}(-1))$. Now define

$$\varphi_{j\, A' \dots B'} = \psi_{j\, A' \dots B'} \pi^{C'} \in C^0(\mathcal{Z}_n(0))$$

and note that $\varphi_{j\, A' \dots B'} \in H^0(\mathcal{Z}_n(0))$ by the isomorphism $H^1(\mathcal{T}(-n-2)) \cong H^0(\mathcal{Z}_n(0))$ proved above. Thus there is a ZRM field $\varphi_{A' \dots B'}$ with

$$\rho_j \varphi_{A' \dots B'} = \varphi_{j\, A' \dots B'} = \psi_{j\, A' \dots B'} \pi^{C'}$$

Now for fixed x we know that $\rho_x f_{ij}$ defines an element of $\mathcal{O}(-n-2)$ over $\mathbb{P}^1$. Therefore $\pi_{A'} \dots \pi_{C'} \rho_x f_{ij}$ is an element of $\mathcal{O}(-1)$ over $\mathbb{P}^1$. Employing Sparling's formula (Example 3.40) we may therefore write

$$\begin{aligned}
\varphi_{j\, A' \dots B'} &= \pi^{C'} \frac{1}{2\pi i} \oint (\xi^{F'} \pi_{F'})^{-1} \xi_{A'} \dots \xi_{C'} \rho_x f_{01}(\omega^A, \xi_{A'}) \xi_{G'} d\xi^{G'} \\
&= \frac{1}{2\pi i} \oint \xi_{A'} \dots \xi_{B'} \rho_x f_{01}(\omega^A, \xi_{A'}) \xi_{C'} d\xi^{C'}
\end{aligned}$$

agreeing with Theorem 6.4.

Remark 6.15. We lacked some rigour in our proof above, failing to mention the subtleties involved in comparing sheaves on different spaces. More complete reasoning requires the use of spectral sequences, which we have not discussed. A full account is given in Ward and Wells [39, §7].

Theorem 6.16.

$$H^1(\mathbb{P}T^+, \mathcal{O}(n-2)) \cong \{\text{ZRM fields } \varphi_{A\ldots B} \text{ of helicity } -n/2 \text{ on } \mathbb{C}M^+\}$$

Proof. This proof has a similar flavour to the previous argument. Define on F^+ the following sheaves

$$\mathcal{K}(n) = \{\text{holomorphic functions } f(x,\pi) \text{ homogeneous of degree } n \text{ in } \pi\}$$

$$\begin{aligned}\mathcal{Q}_A(n+1) = \{&\text{spinor fields } \psi_A(x,\pi) \text{ homogeneous of degree} \\ &(n+1) \text{ in } \pi_{A'} \text{ and satisfying } \pi_{A'}\nabla^{AA'}\psi_A = 0\}\end{aligned}$$

Define a sheaf morphism $D_A : \mathcal{K}(n) \longrightarrow \mathcal{Q}_A(n+1)$ by

$$D_A f = \pi^{A'}\nabla_{AA'} f$$

It is easy to verify that this is well-defined using Lemma 4.48. Moreover it is surjective by the proof of Theorem 4.52. Let $\mathcal{T}(n)$ denote the kernel of D_A and identify as before

$$\begin{aligned}\mathcal{T}(n) = \{&\text{scalar fields } f(x,\pi) \text{ homogeneous of degree } n \\ &\text{in } \pi \text{ which push down to twistor functions}\}\end{aligned}$$

Now we have a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{T}(n) \hookrightarrow \mathcal{K}(n) \xrightarrow{D_A} \mathcal{Q}_A(n+1) \longrightarrow 0$$

whence we obtain a long exact sequence of cohomology

$$\begin{aligned}0 \longrightarrow H^0(F^+, \mathcal{T}(n)) \longrightarrow H^0(F^+, \mathcal{K}(n)) \longrightarrow H^0(F^+, \mathcal{Q}_A(n+1)) \xrightarrow{\delta^*} \\ H^1(F^+, \mathcal{T}(n)) \longrightarrow H^1(F^+, \mathcal{K}(n)) \longrightarrow \ldots\end{aligned}$$

We investigate each of these groups in turn.

- Let $f \in H^0(F^+, \mathcal{T}(n))$. Then we may write

  $$f(x,\pi) = \mu_{A'\ldots B'}(x)\pi^{A'}\ldots\pi^{B'}$$

  where $\mu_{A'\ldots B'}$ is a symmetric holomorphic spinor field on $\mathbb{C}M^+$. The push down condition is

$$\begin{aligned} & \pi^{C'}\pi^{A'}\dots\pi^{B'}\nabla_{CC'}\mu_{A'\dots B'} = 0 \\ \Leftrightarrow \quad & \nabla_{C(C'}\mu_{A'\dots B')} = 0 \end{aligned}$$

Hence we may identify $H^0(F^+, \mathcal{T}(n))$ with the group T_n of $\mu_{A'\dots B'}$ on $\mathbb{C}M^+$ satisfying this equation.

- Let $\lambda \in H^0(F^+, \mathcal{K}(n))$. Then we may write

$$\lambda = \lambda_{A'\dots B'}(x)\pi^{A'}\dots\pi^{B'}$$

where $\lambda_{A'\dots B'}$ is a symmetric holomorphic spinor field on $\mathbb{C}M^+$. There are no additional constraints on $\lambda_{A'\dots B'}$ so we identify $H^0(F^+, \mathcal{K}(n))$ with the group Λ_n of such $\lambda_{A'\dots B'}$.

- Let $\psi_A \in H^0(F^+, \mathcal{Q}_A(n+1))$ and write

$$\psi_A = \psi_{AA'\dots C'}(x)\pi^{A'}\dots\pi^{C'}$$

where $\psi_{AA'\dots C'}$ is a holomorphic spinor field on $\mathbb{C}M^+$ symmetric in its $(n+1)$ primed indices. The defining condition for $\mathcal{Q}_A(n+1)$ gives

$$\begin{aligned} & \pi^{D'}\pi^{A'}\dots\pi^{C'}\nabla^A_{D'}\psi_{A'\dots C'A} = 0 \\ \Leftrightarrow \quad & \nabla^A_{(D'}\psi_{A'\dots C')A} = 0 \end{aligned}$$

We identify $H^0(F^+, \mathcal{Q}_A(n+1))$ with the group Ψ^1_{n+1} of $\psi^A_{A'\dots C'}$ on $\mathbb{C}M^+$ satisfying this equation.

- As in the previous proof, we somewhat unrigorously write $H^1(F^+, \mathcal{T}(n)) = H^1(\mathbb{P}T^+, \mathcal{O}(n))$.

- Recall that $H^1(\mathbb{P}^1, \mathcal{O}(n)) = 0$. Also $\mathcal{K}(n)$ is coherent analytic as a sheaf of sections of the trivial $\mathbb{C}$-bundle over F^+. Using again that $\mathbb{C}M^+$ is Stein, and an appropriate Künneth formula we obtain $H^1(F^+, \mathcal{K}(n)) = 0$.

Rewriting the long exact sequence in our new notation we have the section

$$0 \longrightarrow T_n \hookrightarrow \Lambda_n \xrightarrow{\sigma} \Psi^1_{n+1} \xrightarrow{\delta^*} H^1(\mathbb{P}T^+, \mathcal{O}(n)) \longrightarrow 0 \tag{$\dagger$}$$

where the reader may easily check that σ is given by

$$\sigma(\lambda_{B'\dots C'}) = \nabla^A_{(A'}\lambda_{B'\dots C')}$$

We now relate this sequence to ZRM fields using Hertz potentials, cf. §4.3. Let Φ_{n+2} denote the group consisting of $(n+2)$ unprimed index ZRM fields $\varphi_{A\dots D}$ on $\mathbb{C}M^+$. Define a group homomorphism $P : \Psi^1_{n+1} \longrightarrow \Phi_{n+2}$ by

$$P(\psi_{AB'\dots D'}) = \nabla^{B'}_{(B} \dots \nabla^{D'}_{D} \psi_{A)B'\dots D'}$$

We check that this is well-defined by computing

$$\nabla^{A'}_{A} \nabla^{B'}_{(B} \dots \nabla^{D'}_{D} \psi_{A)B'\dots D'} = \nabla^{B}_{B'} \dots \nabla^{D}_{D'} \nabla^{A}_{(A'} \psi_{B'\dots D')A} = 0$$

which may be verified by expanding out the symmetrisers on each side. Moreover observe that P is surjective. Indeed from Theorem 4.52 we know that given $\varphi_{A\dots D} \in \Phi_{n+2}$ there exists $\psi_{AB'\dots D'}$ defined on $\mathbb{C}M^+$ such that

$$\varphi_{A\dots D} = \nabla^{B'}_{B} \dots \nabla^{D'}_{D} \psi_{AB'\dots D'}$$

and

$$\nabla^{A}_{A'} \psi_{AB'\dots D'} = 0$$

since $\mathbb{C}M^+$ is simply connected and has vanishing second homotopy group. In particular we immediately have $\psi_{AB'\dots D'} \in \Psi^1_{n+1}$ as required.

Finally we claim that $\ker(P) = \mathrm{im}(\sigma)$. For the reverse inclusion we compute

$$\nabla^{B'}_{(B} \dots \nabla^{D'}_{D} \nabla_{A)B'} \lambda_{C'\dots D'} = \frac{1}{2} \epsilon_{(BA} \nabla^{C'}_{C} \dots \nabla^{D'}_{D)} \Box \lambda_{C'\dots D'} = 0$$

The forward inclusion follows from an argument similar to the proof of Theorem 4.52, as articulated in Penrose [29, p. 168].

We therefore have an exact sequence

$$0 \longrightarrow T_n \hookrightarrow \Lambda_n \xrightarrow{\sigma} \Psi^1_{n+1} \xrightarrow{P} \Phi_{n+2}$$

Comparing with (†) we obtain $\Phi_{n+2} \cong H^1(\mathbb{P}T^+, \mathcal{O}(n))$ as required. □

Remark 6.17. Following Hughston and Ward [22, §2.8] we observe that an explicit inverse twistor transform exists in this case. Given a ZRM field $\varphi_{A\dots D}$ let $\psi_{AB'\dots D'}$ be a Hertz potential. We must construct a cover $\{U_j\}$ of $\mathbb{P}T^+$ and twistor functions f_{jk} on U_{jk}. Choose $\{U_j\}$ with the property that

- There exists $Y^\alpha_j \in \mathbb{P}T^+$ such that for all $Z^\alpha \in U_j$ the line joining Y^α_j and Z^α lies entirely in $\mathbb{P}T^+$.

Now suppose $Z^\alpha \in U_j \cap U_k$. Denote by Y_j, Y_k and Z the α-planes in $\mathbb{C}M^+$ corresponding to Y^α_j, Y^α_k and Z^α. Observe by Remark 5.14 that Y_j intersects Z

in a point $p_j \in \mathbb{C}M^+$ defined by the line joining Y_j^α and Z^α in $\mathbb{P}T^+$. Similarly we define $p_k = Y_k \cap Z \in \mathbb{C}M^+$.

We now hypothesise an integral formula for f_{jk}. Let $Z^\alpha = (\omega^A, \pi_{A'})$. Choose an arbitrary contour Γ_{jk} from p_j to p_k lying in Z and define

$$f_{jk}(Z^\alpha) = \int_{\Gamma_{jk}} \psi_{AB'C'\dots D'} \pi^{C'} \dots \pi^{D'} dx^{AB'}$$

We must check that f_{jk} is indendendent of Γ_{jk}, defines a 1-cocycle and reproduces the potential $\psi_{AB'\dots D'}$ under $(\delta^*)^{-1}$. The details are given explicitly in Huggett and Tod [21, p. 96], so we do not reproduce them here.

6.3 The Penrose-Ward Transform

Definition 6.18. Let P be a principal G-bundle over $\mathcal{M}$ with connection $\{A_a\}$. Let E be an associated vector bundle, and D_a the induced covariant derivative. For $U \subset M$ we say that D_a is **integrable** on U iff the parallel transport condition

$$V^a D_a \psi = 0 \text{ for all } V^a \text{ tangent to } U$$

uniquely determines $\psi \in \Gamma(U, E)$ given $\psi(x)$ at any $x \in U$.

Lemma 6.19. Let P be a principal G-bundle over $\mathbb{C}M$ and E an associated vector bundle. Since $\mathbb{C}M$ is contractible, P is trivial so we may work in a single trivialisation. Let A_a denote the gauge connection, F_{ab} its curvature and D_a the induced covariant derivative on E. Then F_{ab} is ASD iff for every α-plane $\tilde{Z}$ we have D_a integrable on $\tilde{Z}$.

Proof. Since $\tilde{Z}$ connected and simply connected, the condition that D_a is integrable on $\tilde{Z}$ is equivalent to stipulating that F_{ab} must vanish on $\tilde{Z}$, i.e.

$$V^a W^b F_{ab} = 0 \text{ for all } V^a, W^a \text{ tangent to } \tilde{Z} \qquad (\star)$$

Indeed geometrically the curvature measures the failure of parallel transport around closed curves in $\tilde{Z}$ to preserve vectors. The integrability condition precisely states that parallel transport of $\psi(x)$ around any curve in $\tilde{Z}$ leaves it unchanged.

Fix an α-plane $\tilde{Z}$. Let $Z^\alpha = (\omega^A, \pi_{A'})$ be the point in P^I defined by $\tilde{Z}$ via the twistor correspondence. By an earlier argument, any vector V^a tangent to $\tilde{Z}$ may be written $V^a = \pi^{A'}\lambda^A$ for some spinor λ^A. Denote by $\varphi_{A'B'}\epsilon_{AB}$ the SD part of F_{ab}. Then $(\star)$ may be written in the form

$$\varphi_{A'B'}\pi^{A'}\pi^{B'} = 0$$

Now varying the α-plane $\tilde{Z}$ we may obtain all $[\pi^{A'}] \in \mathbb{P}^1$, so the vanishing of F_{ab} on all $\tilde{Z}$ is equivalent to the vanishing of the SD part of F_{ab}. $\square$

Theorem 6.20. There is a bijection between

- ASD $GL(n, \mathbb{C})$ gauge fields on $\mathbb{C}M$
- holomorphic rank n vector bundles over P^I, which are trivial when restricted to $\hat{x} \equiv \mathscr{C}(x)$ for all $x \in \mathbb{C}M$.

Proof. Let P be a $GL(n, \mathbb{C})$ bundle over $\mathbb{C}M$ with gauge potential A and ASD curvature F. Let Ψ be the vector bundle associated to the fundamental representation, with covariant derivative D_a. Define a vector bundle E over P^I by choosing the fibre over $Z \in \mathsf{P}^I$ to be

$$E_Z = \{\psi \in \Gamma(\Psi) : V^a D_a \psi = 0 \text{ for all } V^a \text{ tangent to } \tilde{Z} \equiv \mathscr{C}^{-1}(Z)\} \qquad (\dagger)$$

By the previous lemma, since F is ASD we know that D_a is integrable on $\mathbb{C}M$. Therefore each $\psi \in E_Z$ is determined by its value $\psi(x) \in \mathbb{C}^n$ at an arbitrary point $x \in \mathbb{C}M$. In other words, $E_Z \cong \mathbb{C}^n$.

Let U be some simply connected neighbourhood of Z. Make a choice of $y \in \mathbb{C}M$ identifying E_Y with $\mathbb{C}^n$ and varying holomorphically with $Y \in U$. This is possible since the twistor correspondence is appropriately holomorphic. We hence obtain a local trivialisation $U \times \mathbb{C}^n$. The transition functions are clearly holomorphic, so E is a holomorphic vector bundle.

Now let $x \in \mathbb{C}M$ be arbitrary. Choose a vector ψ in the fibre of Ψ over x. Then since F is ASD, ψ determines a section of V on all α-planes through x. Hence ψ determines a section of E restricted to $\hat{x}$. Now choosing n linearly independent vectors at x yields n linearly independent sections of $E|_{\hat{x}}$, which is thus trivial by Remark 3.59.

Conversely let E be a bundle over P^I satisfying the conditions stated above. Define a vector bundle Ψ over $\mathbb{C}M$ by taking the fibre over $x \in \mathbb{C}M$ to be

$$\Psi_x = \Gamma(\hat{x}, E|_{\hat{x}})$$

Since $\hat{x}$ is a Riemann sphere in P^I we see that $\Psi_x \cong \mathbb{C}^n$ by a vector-valued version of Liouville's theorem, cf. Bachman and Narici [2, p. 309]. We endow Ψ with the structure of a vector bundle using the local smoothness of the twistor correspondence.

We must construct a connection A_a on the associated principle bundle to Ψ, or equivalently a covariant derivative D_a on Ψ. This is uniquely determined by specifying how to parallel transport vectors in Ψ_x along curves in $\mathbb{C}M$. It suffices to specify a parallel transport condition for null directions only, for these span the tangent space at each point of $\mathbb{C}M$.

Suppose we are given a null vector V^a at $x \in \mathbb{C}M$, and a vector $\psi(x) \in \Psi_x$. Let $y \in \mathbb{C}M$ be the point defined by moving along V^a from x. We want to define the parallel transport $\psi(y) \in \Psi_y$ of $\psi(x)$. That is to say, we must identify a section of $E|_{\hat{x}}$ with $E|_{\hat{y}}$ in some way. The twistor correspondence provides a natural method.

Write $V^a = \pi^{A'}\lambda^A$ and define a twistor $Z \in \mathsf{P}^I$ by $Z^\alpha = (x^{AA'}\pi_{A'}, \pi_{A'})$. Then $\tilde{Z}$ is an α-plane containing x with V^a as a tangent. Moreover a point $y \in \tilde{Z}$ corresponds to a line $\hat{y} \subset \mathsf{P}^I$ through Z. Now identify sections of $E|_{\hat{x}}$ with sections of $E_{\hat{y}}$ according to their value at Z. This defines a covariant derivative on Ψ.

Observe that by definition the covariant derivative is flat on all α-planes. Hence by the previous lemma the curvature is ASD, as required. It is immediately clear that our constructions are mutually inverse, and the proof is complete. $\square$

Remark 6.21. Although geometrically appealing this proof is useless for practical applications. To equip the reader with tools for calculation, we note some explicit formulae. First observe the subtle point that $\mathsf{P}^I = \mathbb{P}T \setminus \{\pi_{A'} = 0\}$, cf. Huggett and Tod [21, p. 57].

Now we argue that any vector bundle E on P^I satisfying the conditions of the theorem may be trivialised using a cover of just two open sets, namely

$$W_0 = \{(\omega^A, \pi_{A'}) : \pi_{0'} \neq 0\}$$
$$W_1 = \{(\omega^A, \pi_{A'}) : \pi_{1'} \neq 0\}$$

Set $P_0^\alpha = (0,0,0,1) \in W_1$ and $P_1^\alpha = (0,0,1,0) \in W_0$, and denote the corresponding α-planes by $\tilde{P}_0$ and $\tilde{P}_1$. Now let $Z \in W_1$ be an arbitrary twistor. Then $\tilde{P}_1 \cap \tilde{Z}$ is precisely a point P_1^Z. Indeed the intersection is given by the solution of the simultaneous equations

$$\omega^A = ix^{AA'}\pi_{A'}$$
$$0 = ix^{A0'}$$

which is unique in the case $\pi_{1'} \neq 0$.

We claim that $E|_{W_1}$ is trivial. We may assume wlog that E has the form (†). Now trivialise E over W_1 by choosing as coordinates for $\psi \in E_Z$ the value $\psi(P_1^Z) \in \mathbb{C}^n$. The required properties for a local trivialisation are easily checked. Similarly $E|_{W_0}$ is trivial, establishing the result.

Henceforth we fix the cover $\{W_i\}$. Then the structure of E is completely determined by the transition matrix

$$F : W_0 \cap W_1 \longrightarrow GL(n, \mathbb{C})$$

allowing us to explicitly relate the connection A_a on Ψ to the structure of E.

Note that the transition matrix $F(Z)$ is determined by the parallel transport of a vector $\psi(P_0^Z)$ to P_1^Z. That is to say

$$\psi(P_1^Z)^\alpha = F(Z)^{\alpha\beta}\psi(P_0^Z)$$

Work in coordinates where $P_0^Z = x^\mu$ and suppose $P_1^Z = x^\mu + \delta^\mu$. Then by writing the parallel transport condition infinitesimally we produce

$$\psi(P_1^Z)^\alpha = (I^{\alpha\beta} - A_\nu^{\alpha\beta}\delta^\nu)\psi(P_0^Z)^\beta$$

where I is the $n \times n$ identity matrix. For general P_1^Z we break up the path from P_0^Z to P_1^Z into infinitesimal segments and apply this formula, which yields the definition of the path-ordered exponential integral. Hence we may write

$$F(Z) = \mathcal{P}\exp(-\int_\Gamma A_a dx^a)$$

For the inverse transform, suppose we have a transition matrix $F(\omega^A, \pi_{A'})$. Let $G(x, \pi_{A'}) = F(ix^{AA'}, \pi_{A'})$ denote F restricted to a line $\hat{x}$ for some $x \in \mathbb{C}M$. Now E is trivial over $\hat{x}$ so by Lemma 3.26 there exist matrix-valued functions H_i on $W_i \cap \hat{x}$ such that $G = H_0 H_1^{-1}$ on $W_0 \cap W_1 \cap \hat{x}$.

Observe now that every section of $E|_{\hat{x}}$ may be represented by a pair ξ_i of vector fields on $W_i \cap \hat{x}$ where $\xi_0 = H_0\eta_x$ and $\xi_1 = H_1\eta_x$ for some constant $\eta_x \in \mathbb{C}^n$. Letting x vary we obtain a section ψ of Ψ over $\mathbb{C}M$ with $\psi(x) = \eta_x$. Now define the covariant derivative D_a on Ψ by requiring that ψ satisfy the parallel transport equation along null directions $\pi^{A'}\lambda^A$. The associated connection is given by

$$\pi^{A'}A_{AA'} = H_1^{-1}\pi^{A'}\nabla_{AA'}H_1$$

as the reader may verify, cf. Ward and Wells [39, p. 379].

7 Conclusion

It is with mixed feelings that we reach the concluding lines of this review. We have made a long journey across many different terrains, and stop just as the fertile plains open out before us. The reader should now be well-equipped to continue this voyage alone. Here we signpost a few interesting waypoints.

Most obviously, we have failed to give specific applications of the Penrose-Ward transform. Perhaps the most basic nontrivial example considers the minimal coupling of gauge fields to matter. A heuristic overview in provided in Ward and Wells [39, p. 395], and for a full treatment see Hitchin [19].

Twistors have found a particular niche in the study of instantons and monopoles. As a motivational example one might read the "Twistor Quadrille" account of charge quantization in Hughston and Ward [22]. Seminal papers include Hitchin [20] and Atiyah et al. [1].

Twistors are currently being employed as a method of solving nonlinear partial differential equations (PDEs). The philosophy is encapsulated by the Penrose-Ward transform. One represents an system of nonlinear PDEs as compatibility conditions for an overdetermined set of linear PDEs. A recent reference is Dunajski [9].

For the theoretical physicist the most exciting contemporary development is the discovery of twistor string theory by Witten [40]. Certain supersymmetric scattering amplitudes with particularly neat forms in twistor space continue to be explored. It remains to be seen whether the link between twistor theory and string theory is more than just a mathematical curiosity.

Finally, at the other end of the mathematical spectrum, twistor methods admit generalizations to different spacetime signatures. This has yielded various applications in Riemannian geometry, including the study of minimal surfaces. See, for example, Woodhouse [41] or Burstall and Rawnsley [7].

References

[1] M. Atiyah, N. Hitchin, V. Drinfeld, and I. Manin. Construction of Instantons. *Phys. Lett. A*, 65, 1978.

[2] G. Bachman and L. Narici. *Functional Analysis.* Dover, 1966.

[3] T. Bailey, L. Ehrenpreis, and R. Wells. Weak Solutions of the Massless Field Equations. *Proc. R. Soc. Lond. A*, 384, 1982.

[4] V. Bargmann and E. P. Wigner. Group Theoretical Discussion of Relativistic Wave Equations. *PNAS*, 34, 1948.

[5] H. Behnke and K. Stein. Entwicklungen analytischer Funktionen auf Riemannsche Flachen. *Math. Ann.*, 1948.

[6] I. Bialynicki-Birula, E. T. Newman, J. Porter, J. Winicour, B. Lukacs, Z. Perjes, and A. Sebestyen. A Note on Helicity. *J. Math. Phys.*, 22, 1981.

[7] F. Burstall and J. Rawnsley. *Twistor Theory for Riemannian Symmetric Spaces.* Springer, 1990.

[8] P. Di Francesco, P. Mathieu, and D. Senechal. *Conformal Field Theory.* Springer, 1996.

[9] M. Dunajski. *Solitons, Instantons and Twistors.* Oxford University Press, 2010.

[10] M. G. Eastwood, R. Penrose, and R. Wells. Cohomology and Massless Fields. *Commun. Math. Phys.*, 78, 1981.

[11] P. Ehrlich. Null Cones and Pseudo-Riemannian Metrics. *Semigroup Forum*, 43, 1991.

[12] M. Field. *Several Complex Variables and Complex Manifolds II.* Cambridge University Press, 1982.

[13] J. Figueroa-O'Farrill. *Connections on Principal Bundles.* http://empg.maths.ed.ac.uk/Activities/GT/Lect1.pdf, 2006.

[14] F. Forstnerič. *Stein Manifolds and Holomorphic Mappings.* Springer, 2011.

[15] R. Geroch. Asymptotic structure of space-time. In P. Esposito and L. Witten, editors, *Symposium on the Asymptotic Structure of Space-time*, pages 1–105. Plenum, 1977.

[16] P. Griffiths and J. Harris. *Principles of Algebraic Geometry.* Wiley, 1978.

[17] R. Hartshorne. *Algebraic Geometry.* Springer, 1977.

[18] R. Hirzebruch. *Topological Methods in Algebraic Geometry.* Springer, 1978.

[19] N. Hitchin. Linear Field Equations on Self-Dual Space. 370, 1980.

[20] N. Hitchin. Monopoles and Geodesics. 83, 1982.

[21] S. Huggett and K. Tod. *An Introduction To Twistor Theory.* Cambridge University Press, 1994.

[22] L. P. Hughston and R. S. Ward. *Advances in Twistor Theory.* Pitman Advanced Pub. Program, 1979.

[23] C. J. Isham. *Modern Differential Geometry for Physicists.* World Scientific Publising, 1999.

[24] A. Jadcyzk. On Conformal Infinity and Compactifications of Minkowski Space. *Advances in Applied Clifford Algebras*, 21, 2011.

[25] F. John. The Ultrahyperbolic Differential Equation with Four Independent Variables. *Duke Math. J.*, 4, 1938.

[26] J. Morrow and K. Kodaira. *Complex Manifolds.* AMS Chelsea, 1971.

[27] G. L. Naber. *Topology, Geometry and Gauge Fields.* Springer, 2011.

[28] M. Nakahara. *Geometry, Topology and Physics.* Taylor & Francis, 2003.

[29] R. Penrose. Zero Rest-Mass Fields Including Gravitation: Asymptotic Behaviour. *Proc. R. Soc. Lond. A*, 284, 1965.

[30] R. Penrose. Twistor Algebra. *J. Math. Phys.*, 8, 1967.

[31] R. Penrose. On the Origins of Twistor Theory. In *Gravitation and Geometry, a Volume in Honour of I. Robinson.* Biblpolis, Naples, 1987.

[32] R. Penrose and M. MacCallum. Twistor Theory: an Approach to the Quantization of Fields and Spacetime. *Phys. Rept.*, 6, 1972.

[33] R. Penrose and W. Rindler. *Spinors and Space-Time. Vol I: Two-Spinor Calculus and Relativistic Fields.* Cambridge University Press, 1984.

[34] R. Penrose and W. Rindler. *Spinors and Space-Time. Vol II: Spinor and Twistor Methods and Space-Time Geometry.* Cambridge University Press, 1986.

[35] J. H. Sampson and G. Washnitzer. A Künneth Formula for Coherent Algebraic Sheaves. *Illinois J. Math.*, 3, 1959.

[36] N. Steenrod. *The Topology of Fibre Bundles.* Oxford University Press, 1951.

[37] H. Urbantke. The Hopf Fibration - Seven Times in Physics. *J. Geom. Phys.*, 46, 2003.

[38] R. Ward. On Self-Dual Gauge Fields. *Phys. Lett. A*, 61, 1977.

[39] R. Ward and R. Wells. *Twistor Geometry and Field Theory.* Cambridge University Press, 1990.

[40] E. Witten. Perturbative Gauge Theory as a String Theory in Twistor Space. *Commun. Math. Phys.*, 252, 2004.

[41] N. Woodhouse. Contour Integrals for the Ultrahyperbolic Wave Equation. *Proc. Math. Phys.*, 438, 1992.